Mechanics Based Problems

1. In the previous lesson, you wrote each of the following systems in matrix form. First, find the eigenvalues and associated eigenvectors of each of the systems. Then, write the general solution of the given system:

(a) $\frac{dx}{dt} = x + 2y$

$\frac{dy}{dt} = 4x + 3y$

$$\vec{X} = C_1\begin{bmatrix}1\\2\end{bmatrix}e^{5t} + C_2\begin{bmatrix}-1\\1\end{bmatrix}e^{-t}$$

(b) $\frac{dx}{dt} = 3x - y$

$\frac{dy}{dt} = 9x - 3y$

$$\vec{X} = C_1\begin{bmatrix}1+3t\\9t\end{bmatrix} + C_2\begin{bmatrix}-t\\1-3t\end{bmatrix}$$

(c) $\frac{dx}{dt} = 6x - y$

$\frac{dy}{dt} = 5x + 2y$

$$\vec{X} = C_1\left(\begin{bmatrix}2/5\\1\end{bmatrix}\cos t - \begin{bmatrix}1/5\\0\end{bmatrix}\sin t\right)e^{4t} + C_2\left(\begin{bmatrix}1/5\\0\end{bmatrix}\cos t + \begin{bmatrix}2/5\\0\end{bmatrix}\sin t\right)e^{4t}$$

2. For the previous problems, verify your solutions.

(a)

– INSERT YOUR PROOF HERE –

$$C_1\begin{bmatrix}5\\0\end{bmatrix}e^{5t}+C_2\begin{bmatrix}1\\-1\end{bmatrix}e^{-t}=C_1\begin{bmatrix}5\\0\end{bmatrix}e^{5t}+C_2\begin{bmatrix}1\\-1\end{bmatrix}\quad \text{QED}$$

(b)

– INSERT YOUR PROOF HERE –

$$\begin{bmatrix}3C_1-C_2\\9C_1-3C_2\end{bmatrix}=\begin{bmatrix}3C_1-C_2\\9C_1-3C_2\end{bmatrix}\quad \text{QED}$$

(c)

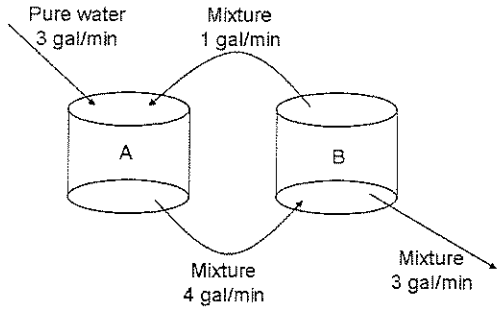
– INSERT YOUR PROOF HERE
(RECOMMEND USING MMA 6.0)

$$\vec{X}'=A\vec{X}$$

OUT = TRUE QED

Problem Solving Problems

1. Recall the first mixing problem from Lesson 49:
A tank contains 50 gallons of water in which 25 pounds of salt are dissolved. A second tank, B, contains 50 gallons of pure water. Liquid is pumped in and out of the tanks at rates shown in the figure below.



(a) Using your work from Lesson 49, find the eigenvalues and eigenvectors of of the coefficient matrix.

$\lambda_1 = -3/25$, $\lambda_2 = -1/25$

$\vec{K}_1 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$, $\vec{K}_2 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

(b) Write the general solution to the system of differential equations.

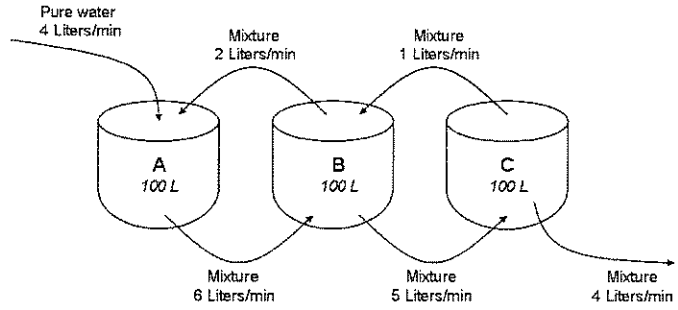
$$\vec{X} = c_1 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} e^{-3/25\,t} + c_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} e^{-1/25\,t}$$

(c) Verify that your solution satisfies the system of differential equations.

- INSERT YOUR PROOF HERE -

$$\begin{bmatrix} -\frac{3}{2}e^{-3t/25} - \frac{1}{2}e^{-t/25} \\ 3e^{-3t/25} - e^{-t/25} \end{bmatrix} = \begin{bmatrix} \frac{3}{2}e^{-3t/25} - \frac{1}{2}e^{-t/25} \\ 3e^{-3t/25} - e^{-t/25} \end{bmatrix}$$ QED

2. Recall the second mixing problem from Lesson 49:
Using the information in the figure below, derive the system of differential equations describing the number of mg of salt at any time in tanks A, B, and C, respectively.



(a) Using your work from Lesson 49, find the eigenvalues and eigenvectors of of the coefficient matrix.

$\lambda_1 = -.1054$, $\lambda_2 = -.0532$, $\lambda_3 = -.0214$

$\vec{K}_1 = \begin{bmatrix} .488 \\ -1.108 \\ 1 \end{bmatrix}$, $\vec{K}_2 = \begin{bmatrix} -.184 \\ -.063 \\ 1 \end{bmatrix}$, $\vec{K}_3 = \begin{bmatrix} .296 \\ .572 \\ 1 \end{bmatrix}$

(b) Write the general solution to the system of differential equations.

$$\vec{X} = c_1 \vec{K}_1 e^{\lambda_1 t} + c_2 \vec{K}_2 e^{\lambda_2 t} + c_3 \vec{K}_3 e^{\lambda_3 t}$$

(c) Verify that your solution satisfies the system of differential equations.

< INSERT YOUR PROOF HERE >

Using Mma $\Rightarrow$ $\vec{X}'[t] = A\vec{X}[t] \Rightarrow$ TRUE

QED