Interval Estimation for Normal-based Proportions: With Application to Armor Testing

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Motivation

- N shots are fired into Armor Type 1
- Penetrations are recorded as $X_1, \ldots, X_N$
- "In future tests against a different armor, we want some assurance that at least one of the shots will have had the capability to penetrate a certain thickness of Armor Type 1."

Introduction

What we seek is a lower limit on the probability that at least one out of k future shots has X>L, where

- X is the penetration into Armor Type I of a randomly selected experimental unit (for a given threat)
- L is a specified armor thickness

Introduction

What we want is a lower limit on …

$$
\begin{aligned}
&P\left(\text{"at least 1 of k shots has } X > L\text{"}\right) \\
&= 1 - P\left(\text{"no shots have } X > L\text{"}\right) \\
&= 1 - P\left(\text{"all shots have } X \le L\text{"}\right) \\
&= 1 - \left[P\left(X \le L\right)\right]^k
\end{aligned}
$$

or, equivalently, an upper limit on

$$\left[P\left(X \le L\right)\right]^k$$

Introduction

or, since the power function is decreasing on (0,1), we only need to consider an upper limit on

$$P\left(X \leq L\right) = \theta$$

Introduction

Treat this as a problem of estimating either …

- a binomial proportion, or
- a normal probability (historical precedent has shown penetrations to be normally distributed)

Wilson's score interval (1927)

$$\frac{\hat{p} + \frac{1}{2n} z_{1-\alpha}^2 + z_{1-\alpha}\sqrt{\frac{1}{n}\hat{p}\left(1-\hat{p}\right) + \frac{1}{4n^2} z_{1-\alpha}^2}}{1 + \frac{1}{n} z_{1-\alpha}^2}$$

where $\hat{p} = \frac{\#\text{ of } X \le L}{n}$

Owen & Hua's exact method (1977)

1) Based upon non-central t distribution
2) Solution requires an iterative numerical algorithm
3) Results are usually presented tabularly (e.g. Odeh & Owen [1980]), for specific values of

$$n, 1-\alpha, \text{ and } K\left(=\frac{\bar{x}-L}{s}\right),$$

often necessitating interpolation

Upper bound for a normal probability

100(1-α)% upper confidence limit is

$$\Phi\left(\frac{\delta}{\sqrt{n}}\right)$$

where δ is the value of the non-centrality parameter which satisfies

$$H\left(\frac{\bar{x}-L}{s/\sqrt{n}};n-1,\delta\right)=P\left(T_{n-1,\,\delta}\le\frac{\bar{x}-L}{s/\sqrt{n}}\right)=1-\alpha$$

Upper bound for a normal probability

Initial estimate

$$\delta_1 = \frac{\bar{x} - L}{s/\sqrt{n}} - \Phi^{-1}(1-\alpha)$$

Updated estimates

$$\delta_{r+1} = \delta_r - \frac{H\left(\frac{\bar{x} - L}{s/\sqrt{n}}; n-1, \delta_r\right) - (1-\alpha)}{\frac{dH}{d\delta}\left(\frac{\bar{x} - L}{s/\sqrt{n}}; n-1, \delta_r\right)}$$

where the derivative of H in the updated estimate formula equals …

$$\frac{s}{\bar{x} - L}\sqrt{\frac{2(n-1)}{n}}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\left[H\left(\sqrt{\frac{n-2}{n-1}}\,\frac{\bar{x} - L}{s/\sqrt{n}}; n-2, \delta_r\right) - H\left(\sqrt{\frac{n}{n-1}}\,\frac{\bar{x} - L}{s/\sqrt{n}}; n, \delta_r\right)\right]$$

Upper bound for a normal probability

Convergence to a solution is very quick – accuracy to 8th decimal place usually in 4-7 iterations. However, programming this algorithm is messy.

Upper bound for a normal probability

If we rewrite the parameter of interest as

$$\theta = P\left(X \le L\right) = \Phi\left(\frac{L-\mu}{\sigma}\right)$$

We see that the confidence interval we are trying to construct is for a function of the armor thickness, L, and the unknown parameters, μ and σ. Problems of this nature have been solved using generalized inference.

Upper bound for a normal probability

Generalized inference

(Tsui & Weerahandi, 1989;

Weerahandi, 1993)

1) **Monte-Carlo method**
2) **Requires construction of generalized pivot for the parameter of interest**

Generalized Pivots

Notation:

X = Random variable, possibly a vector, often a set of sufficient statistics

x = Observed value of X

ξ = Parameter vector

$\theta = f(\xi)$ = Parameter of interest

Generalized Pivots

Let R(X; x, ξ) be a function satisfying the following two conditions:

1) The distribution of R is free of any unknown parameters.

2) The observed value of R (denoted by r) depends on ξ only through θ; i.e., $r = \theta$.

Then R is said to be a generalized pivot for θ. The percentiles of R are used to form confidence intervals/limits for θ.

Consider the following random variable:

$$R = R\left(\bar{X}, S; \bar{x}, s, \mu, \sigma\right)$$

$$= \Phi\left(\frac{1}{\sqrt{n}} \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} + \frac{L - \bar{x}}{s} \frac{1}{\sqrt{n-1}} \sqrt{\frac{(n-1) S^2}{\sigma^2}}\right)$$

$$\overset{dist.}{\equiv} \Phi\left(\frac{Z}{\sqrt{n}} + \frac{L - \bar{x}}{s} \sqrt{\frac{V}{n-1}}\right)$$

$N(0,1)$ $\qquad$ χ^2_{n-1}

The distribution of R does not depend on the unknown parameters. So Condition 1 is satisfied.

The observed value of R is

$$
\begin{aligned}
r &= r\left(\bar{x}, s; \bar{x}, s, \mu, \sigma\right) \\
&= \Phi\left(\frac{1}{\sqrt{n}} \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} + \frac{L-\bar{x}}{s} \frac{1}{\sqrt{n-1}} \sqrt{\frac{(n-1)s^2}{\sigma^2}}\right) \\
&= \Phi\left(\frac{\bar{x}-\mu}{\sigma} + \frac{L-\bar{x}}{\sigma}\right) \\
&= \Phi\left(\frac{L-\mu}{\sigma}\right)
\end{aligned}
$$

The observed value of R equals the parameter of interest.
Condition 2 is satisfied, and so R is a generalized pivot.

Pivot Development

To obtain a 100(1-α)% upper limit on

$$P\left(X \le L\right) = \Phi\left(\frac{L-\mu}{\sigma}\right)$$

take the 100(1-α)th percentile of R using Monte Carlo simulation.

Generalized Algorithm

E.g., Monte Carlo simulation for 95th percentile of *R*

1) **Set L**

2) **Calculate** $\bar{x}, s$ **from observed data** $x_1, x_2, \ldots, x_n$

3) **Generate m samples from** $Z \sim N(0,1)$, $V \sim \chi^2_{n-1}$ **where m is "large", say m>1000.**

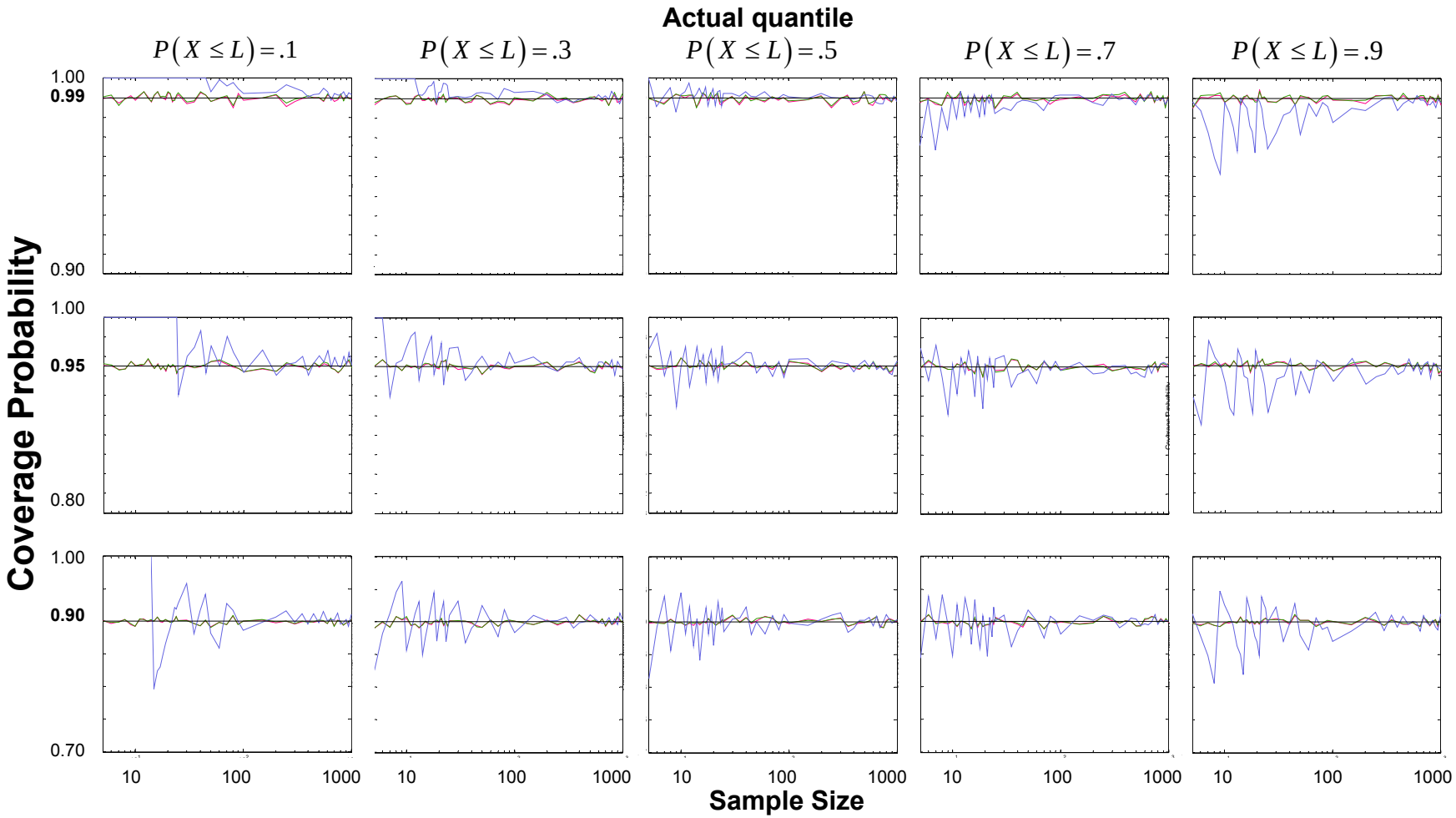
4) **Let** $$R = \Phi\left(\frac{Z}{\sqrt{n}} + \frac{L-\bar{x}}{s}\sqrt{\frac{V}{n-1}}\right)$$

5) **Sort the values of *R*.**

6) **The 95% upper limit for** $P(X \le L)$ **is *R*([.95m])**

Coverage Probabilities

Comparison of coverage probabilities between Wilson's score, O&H exact and generalized limits.



Nominal coverage probability shown by horizontal solid black line.

(4000 replications, 2500 values of R)

Comments on Coverage Probabilities

Wilson's score limits

- Coverage oscillates wildly about nominal level due to discreteness of binomial distribution

Owen & Hua exact limits

- Coverage is good for all sample sizes and quantile probabilities.

Generalized limits

- Coverage is good for all sample sizes and quantile probabilities; tracks nearly identically with O&H exact limits.
- Hannig gives regularity conditions which guarantee asymptotic convergence to nominal confidence level.

Application to Armor Testing

31 penetrations are recorded from an armor test. The sample mean and standard deviation are 20.25 and 3.09, respectively. Also, 26 penetrations are <23; remaining 5 are >23.

What is the probability that at least 1 of 5 future shots will have a penetration value of at least 23 units?

	Generalized	O&H's	Wilson's
95% UL for P(X<23) =	.892	.892	.919
95% LL for P(at least 1 of 5 X>23) =	.435	.436	.344

"We can be 95% confident that the probability of one or more penetrations exceeding 23 units when 5 rounds are fired is at least 43.5%."

"We need to fire 15 rounds to be 95% confident that the probability of one or more penetrations exceeding 23 units is at least 80%."

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