Statistics of non-real zeros and critical points of real random polynomials

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What is a real random polynomial?

Statistics of non-real zeros and critical points of real random polynomials

What is a real random polynomial?

$$f_N(x) = \sum_{k=0}^{N} a_k x^k$$

real random variable

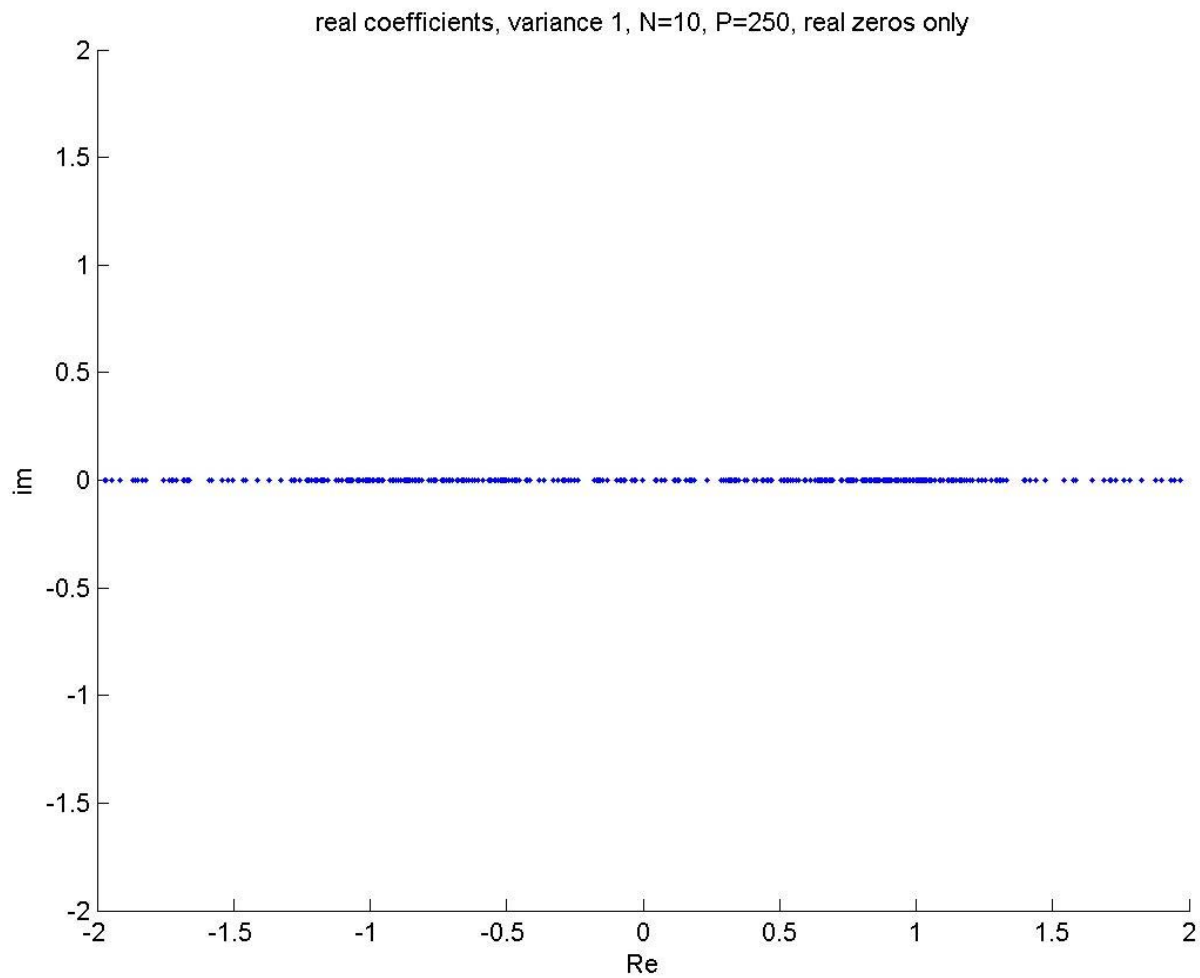
We will always assume the coefficients are independent.

Example R1

$$f_N(x) = \sum_{k=0}^{N} a_k x^k$$

a_k is real standard Gaussian random variable

$N = 10$
$\#$ of polynomials $= 250$

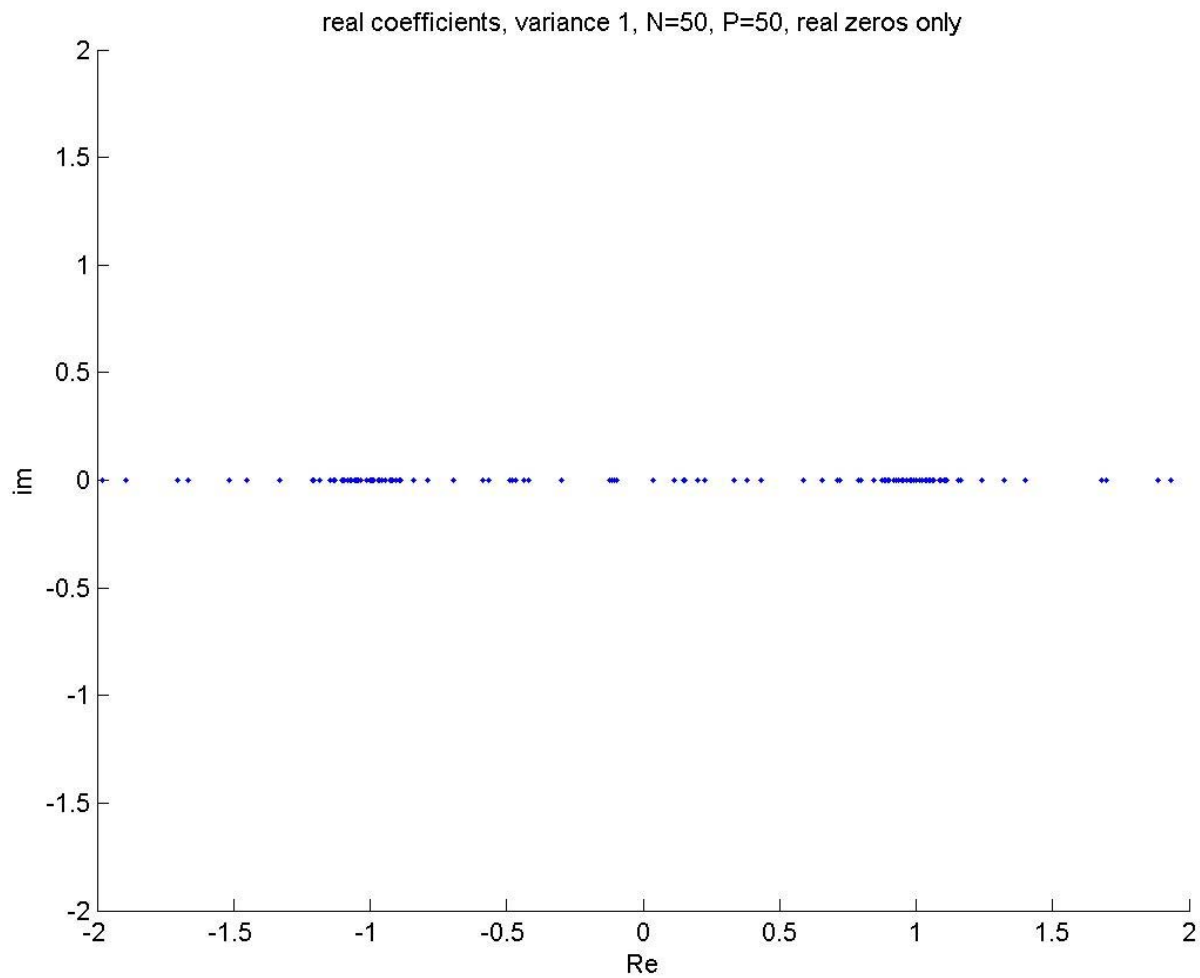


Example R1

$$f_N(x) = \sum_{k=0}^{N} a_k x^k$$

a_k is real standard Gaussian random variable

$N = 50$
$\#$ of polynomials $= 50$



Example R2

$$f_N(x) = \sum_{k=0}^{N} a_k x^k \qquad a_k \text{ is real Gaussian random variable with variance } \binom{N}{k}$$

$$f_N(x) = \sum_{k=0}^{N} a_k \binom{N}{k}^{\frac{1}{2}} x^k \qquad a_k \text{ is standard real Gaussian random variable}$$

Why this polynomial?

- Probably the more natural definition (Edelman-Kostlan)

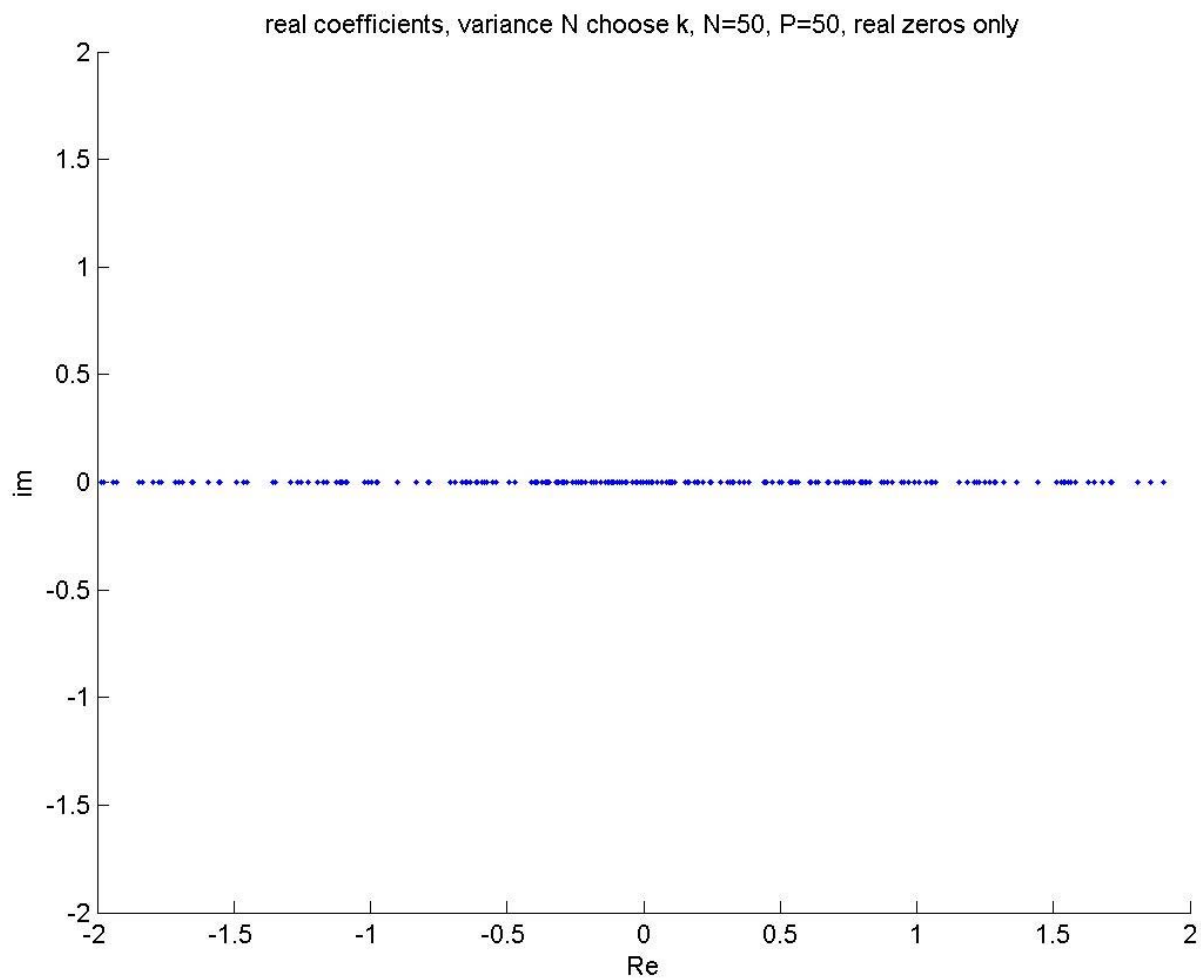
- Represents chaotic spin wavefunctions that arise in physics (Majorana)

- It has nice invariance properties.

Example R2

$$f_N(x) = \sum_{k=0}^{N} a_k \binom{N}{k}^{\frac{1}{2}} x^k \quad a_k \text{ is standard real Gaussian random variable}$$

$N = 10$
$\#$ of polynomials $= 250$

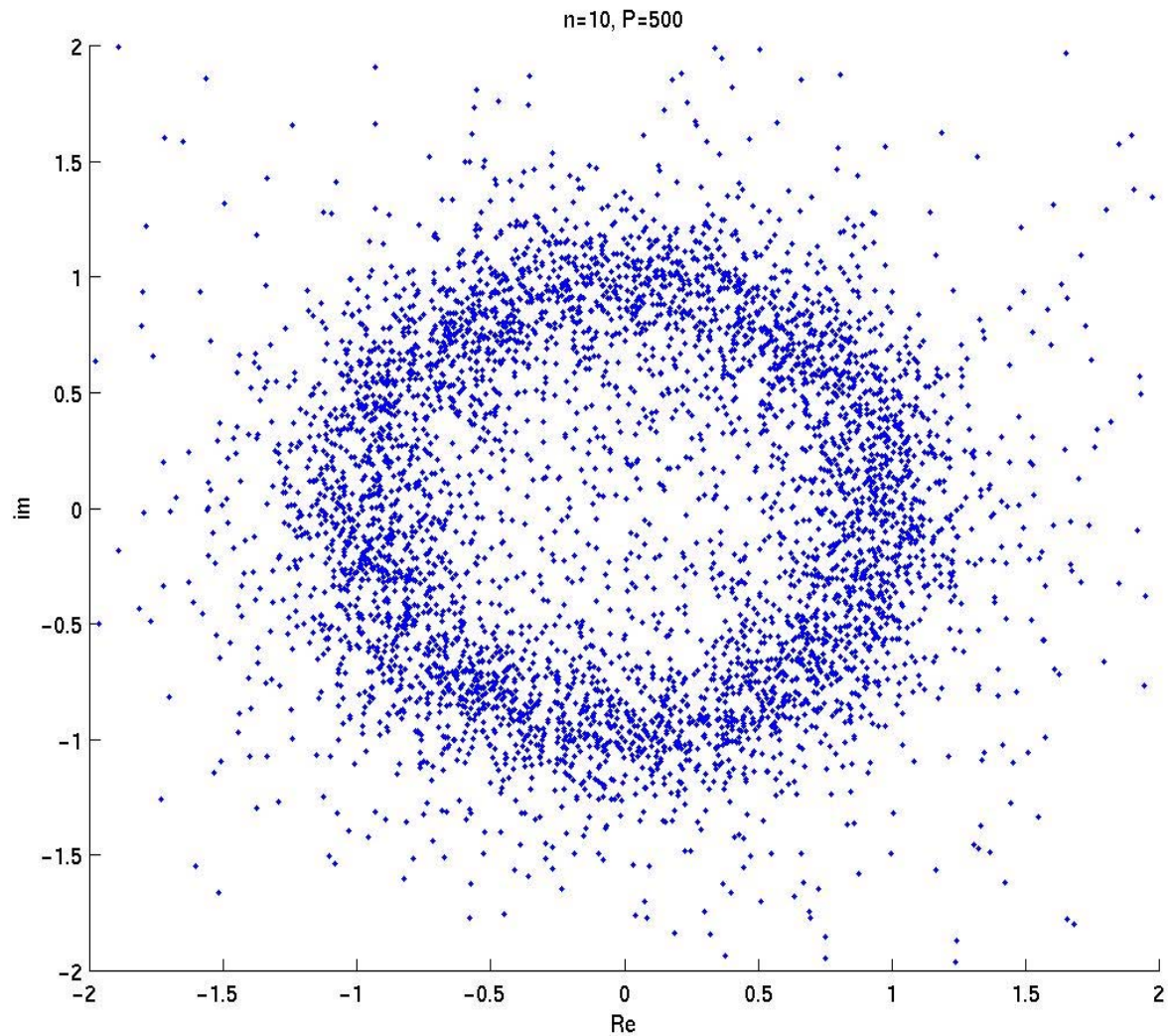


Example C1

$$f_N(z) = \sum_{k=0}^{N} c_k z^k$$

c_k is complex standard Gaussian random variable

$N = 10$
$\#$ of polynomials $= 500$
Total number of zeros $= 5000$



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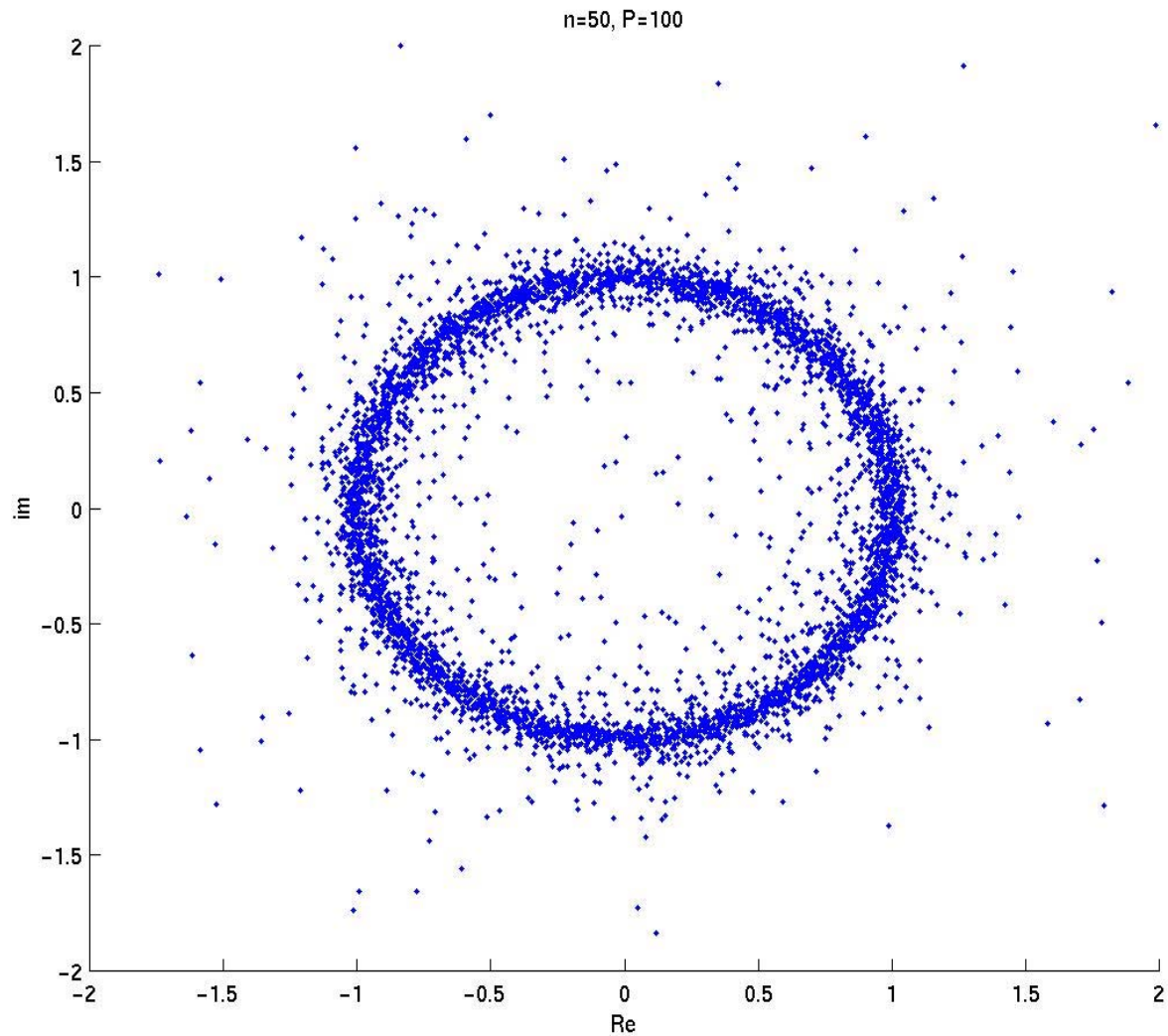

Example C2

$$f_N(z) = \sum_{k=0}^{N} c_k \binom{N}{k}^{\frac{1}{2}} z^k$$

c_k is complex standard Gaussian random variable

Why this polynomial?

· Same answers as Example R2.

· Also, $\binom{N}{k}^{\frac{1}{2}} z^k$ is an orthonormal basis with respect to the usual metric on the Riemann sphere.

(it arises naturally)

Example C2

$$f_N(z) = \sum_{k=0}^{N} c_k \binom{N}{k}^{\frac{1}{2}} z^k$$

c_k is complex standard Gaussian random variable

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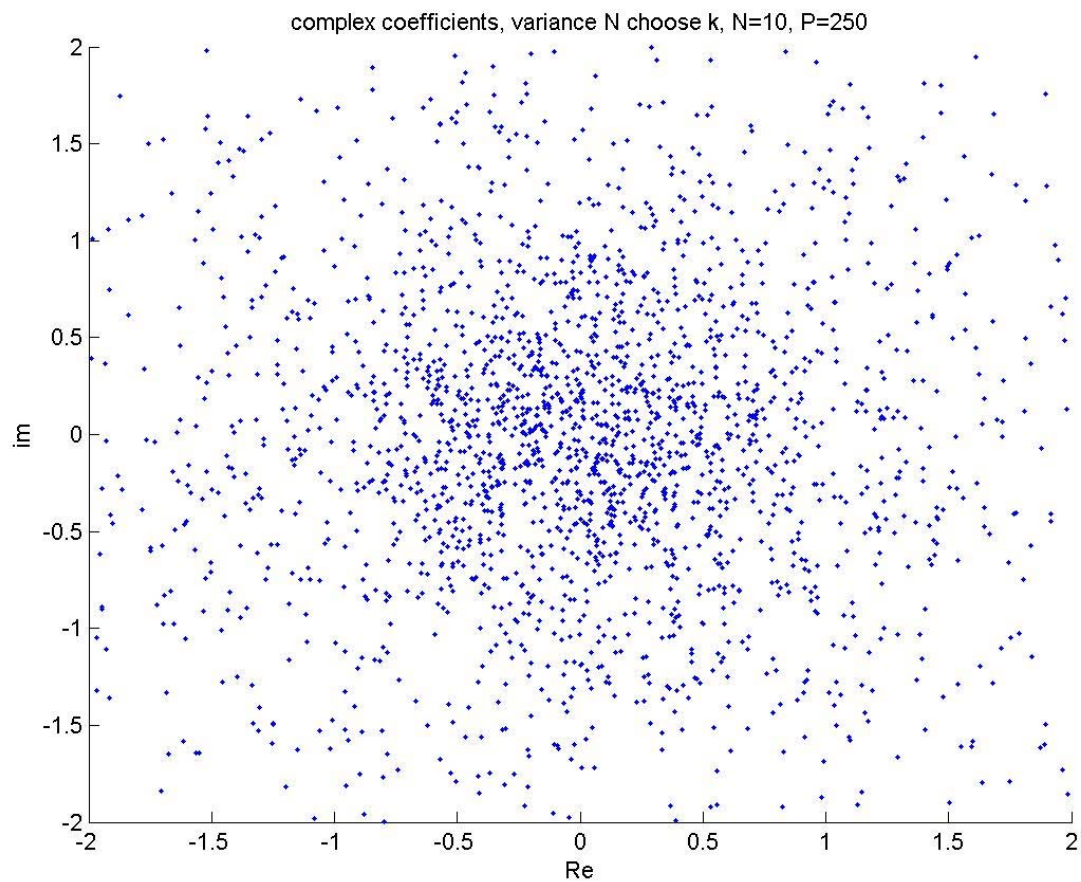

Example C2

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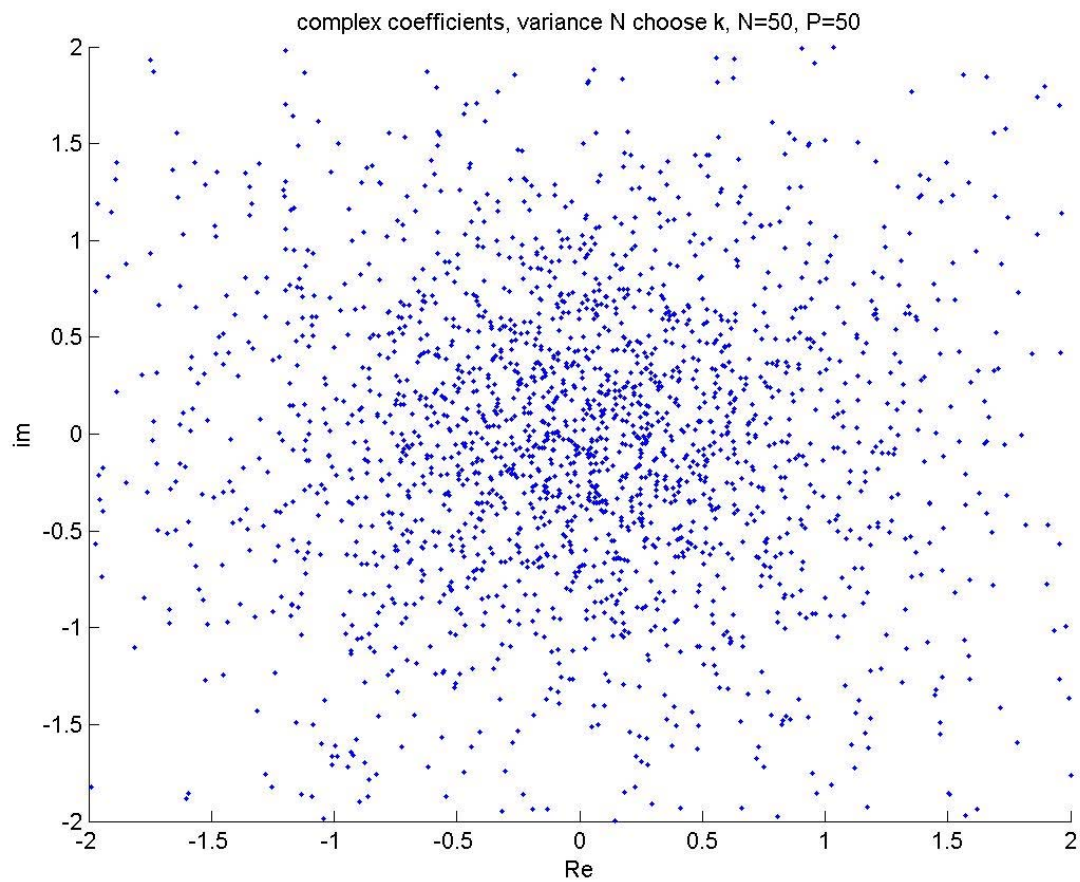

Example C2
Plotting Zeros on the Sphere

$$f_N(z) = \sum_{k=0}^{N} c_k \binom{N}{k}^{\frac{1}{2}} z^k \qquad c_k \text{ is complex standard Gaussian random variable}$$

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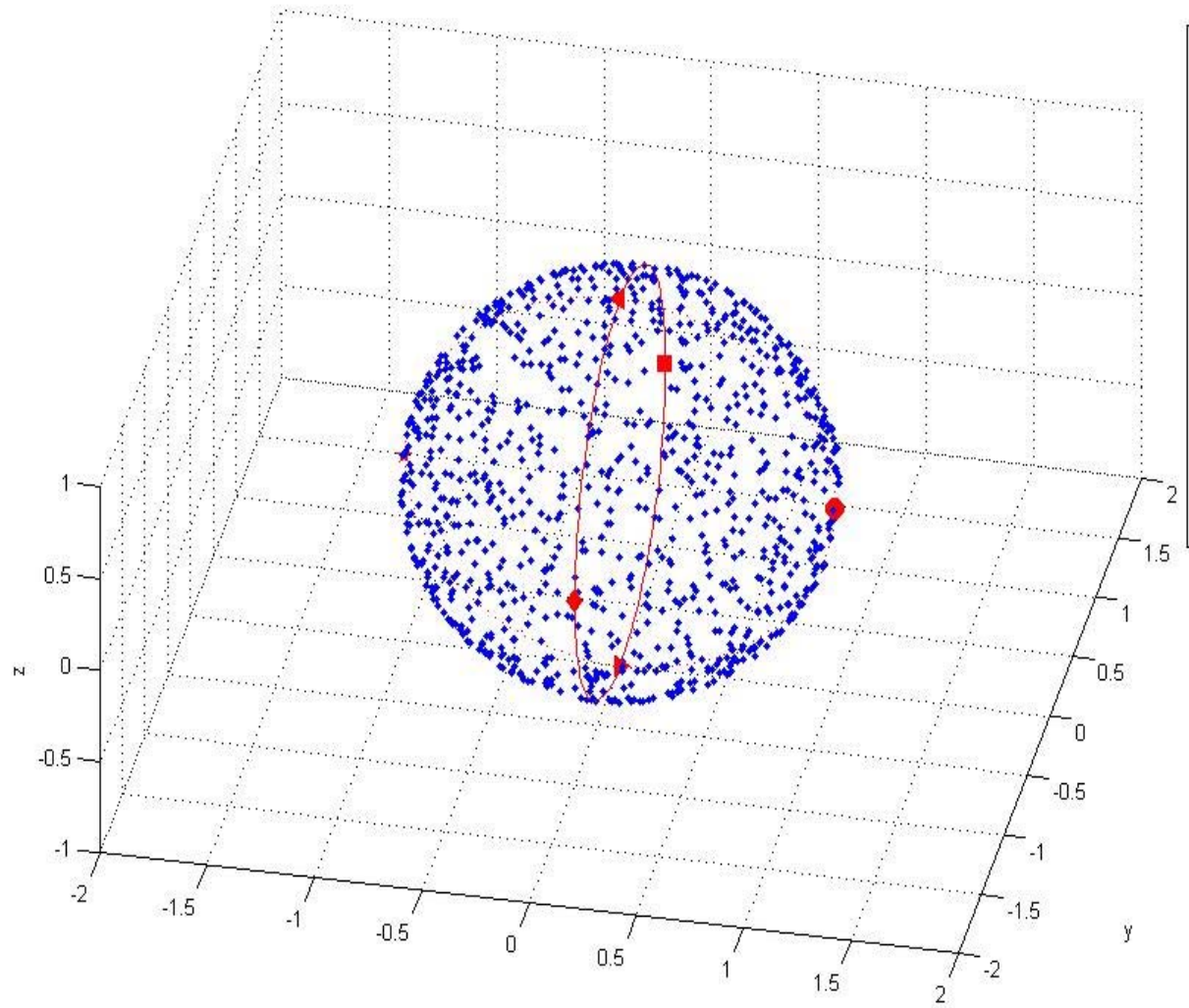


What questions can we ask?

Distribution of zeros, correlation between zeros
(Kac, Hammersley, Edelman-Kostlan, etc.)

Expected number of real zeros (Kac, Rice, Shub-Smale, etc.)

Probability of no real zeros (Dembo-Poonen-Shao-Zeitouni, Li-Shao)

Hole probability (Sodin-Tsirelson, Zrebiec)

Higher Moments and Asymptotic Normality of (Z_f, ϕ)
(Sodin-Tsirelson, Shiffman-Zelditch)

Generalizations to real and complex manifolds
(Shiffman-Zelditch, Bleher-Shiffman-Zelditch, Douglas-Shiffman-Zelditch)

Distribution of zeros

$$f_N(x) = \sum_{k=0}^{N} a_k x^k$$

a_k is real standard Gaussian random variable

Let $Z_{f_{N,a}(x)} = \sum_{f_{N,a}(x_j)=0} \delta_{x_j}$

$E(Z_f) =$ density of zeros

Note: $E(\cdot)$ here denotes integration w.r.t the standard real Gaussian measure on $\mathbb{R}^{N+1}$

Kac found a formula for $E(Z_f)$:

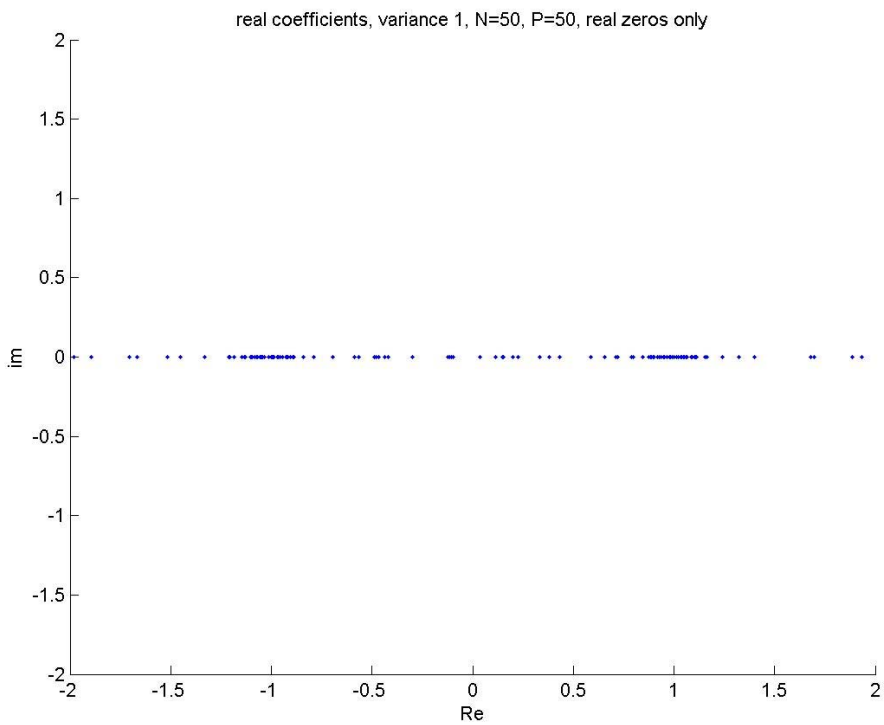
$$E(Z_{f_N(x)}) = \frac{1}{\pi}\sqrt{\frac{1}{x^2-1} - \frac{(N+1)^2 x^{2N}}{(x^{2N+2}-1)^2}}$$

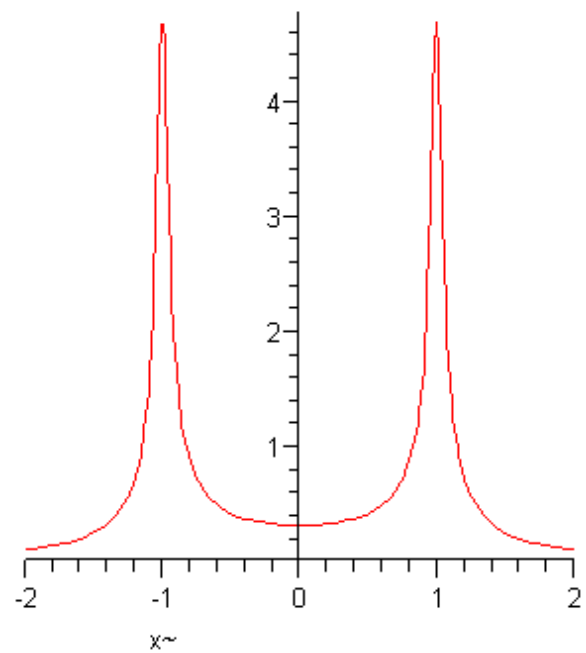
Example R1

$$f_N(x) = \sum_{k=0}^{N} a_k x^k$$

a_k is real standard Gaussian random variable

$$E(Z_{f_N(x)}) = \frac{1}{\pi}\sqrt{\frac{1}{x^2-1} - \frac{(N+1)^2 x^{2N}}{(x^{2N+2}-1)^2}}$$

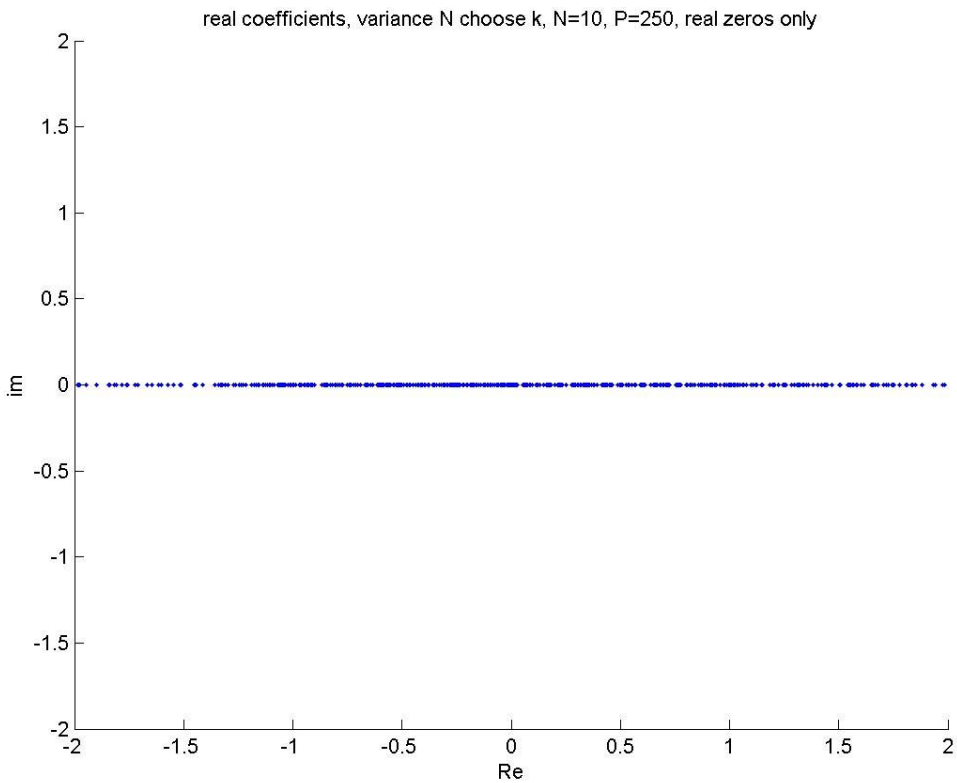


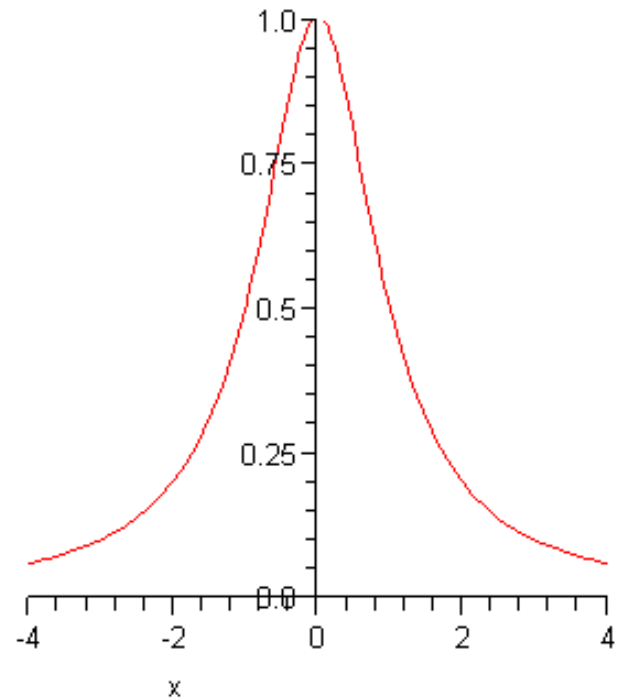


Example R2

$$f_N(x) = \sum_{k=0}^{N} a_k \binom{N}{k}^{\frac{1}{2}} x^k \quad a_k \text{ is real standard Gaussian random variable}$$

$$E(Z_{f_N(x)}) = \frac{\sqrt{N}}{\pi} \frac{1}{1+x^2}$$



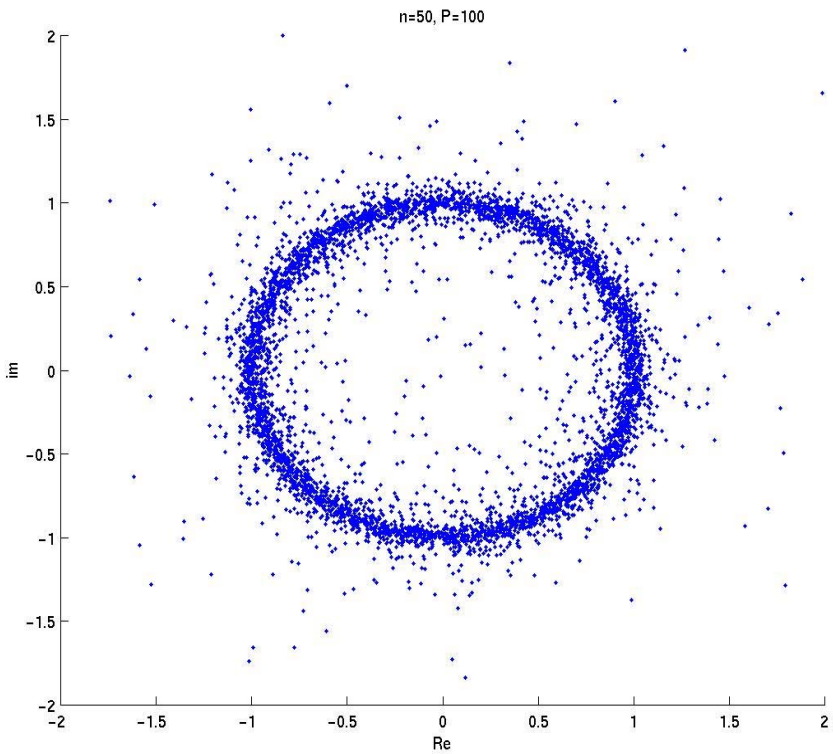


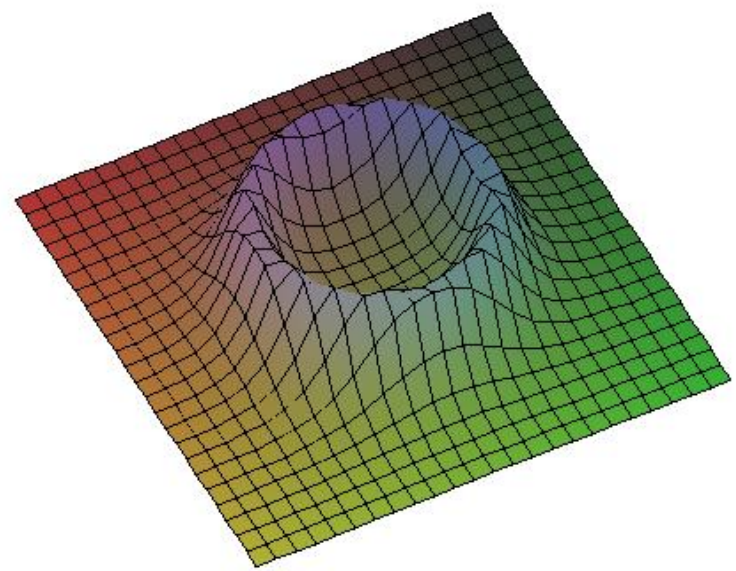
Example C1

$$f_N(z) = \sum_{k=0}^{N} c_k z^k$$

c_k is complex standard Gaussian random variable

$$E(Z_{f_N(z)})$$



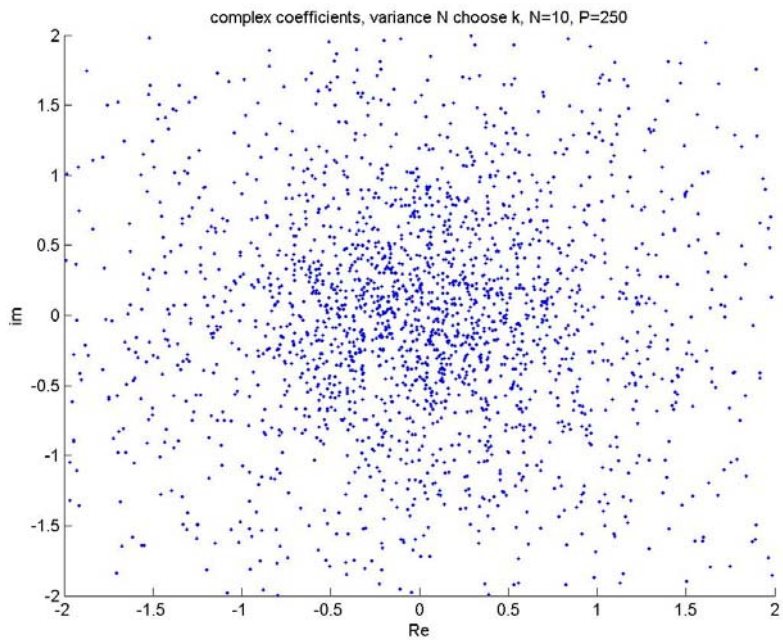


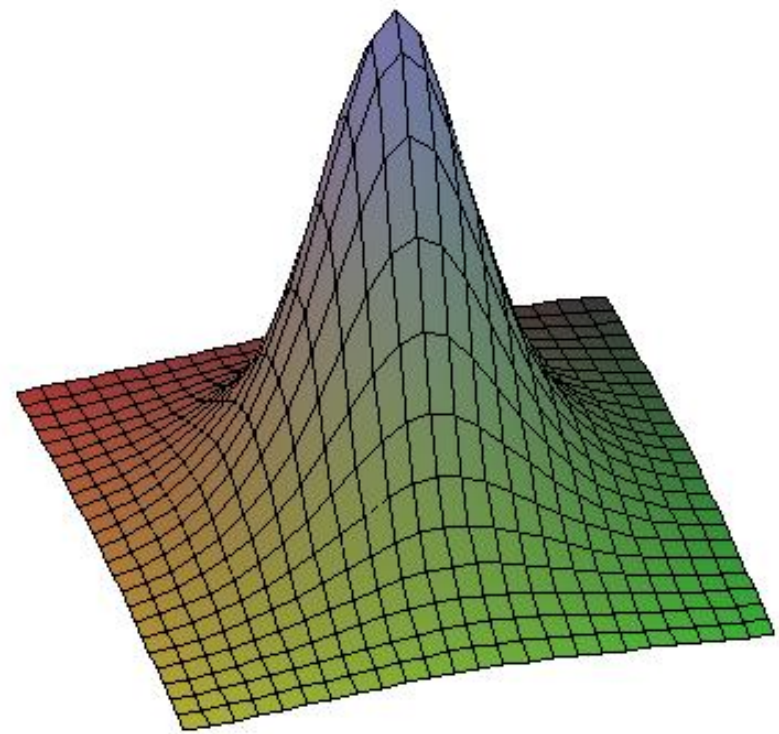
Example C2

$$f_N(z) = \sum_{k=0}^{N} c_k \binom{N}{k}^{\frac{1}{2}} z^k$$

c_k is complex standard Gaussian random variable

$$E(Z_{f_N(z)}) = \frac{N}{\pi} \frac{1}{(1+|z|^2)^2}$$

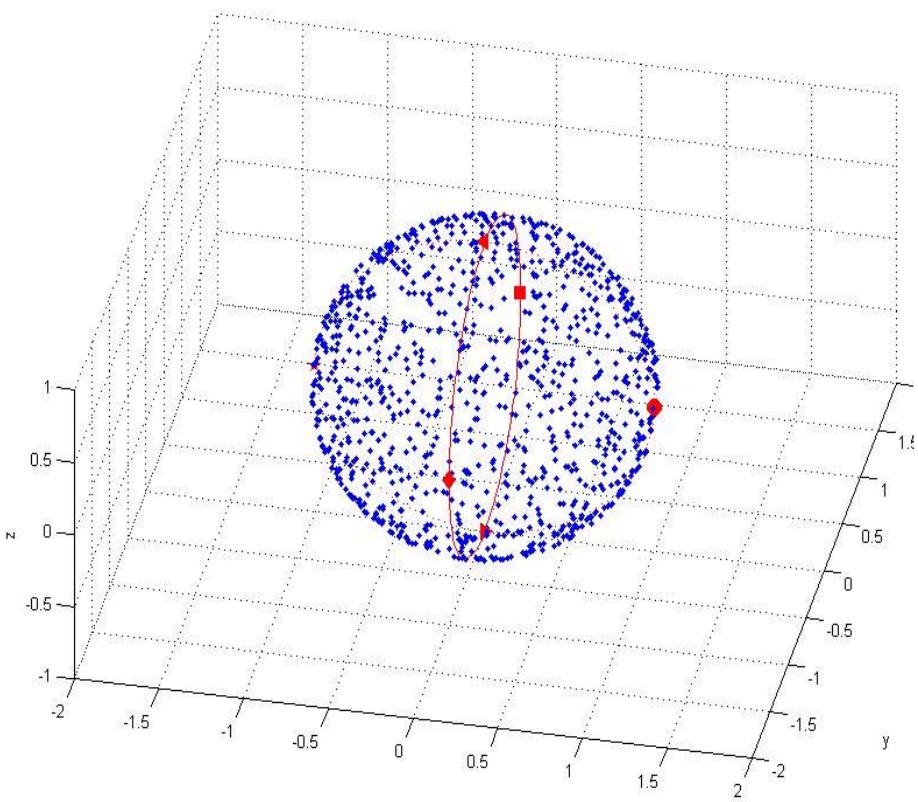




Example C2
Plotting zeros on the sphere

$$f_N(z) = \sum_{k=0}^{N} c_k \binom{N}{k}^{\frac{1}{2}} z^k \qquad c_k \text{ is complex standard Gaussian random variable}$$

$$E(Z_{f_N(z)}) = \frac{N}{\pi} \frac{1}{(1+|z|^2)^2}$$



$E(Z_f)$ is constant on the unit sphere (constant with respect to the usual metric on S^2)

What questions can we ask?

What about the complex zeros of real polynomials?

complex

$$f_N(z) = \sum_{k=0}^{N} a_k z^k$$

real

Why?

· Quantum chaos (Prosen).

· Potential to get information about real zeros (Zelditch)

· It is an interesting mathematical problem.

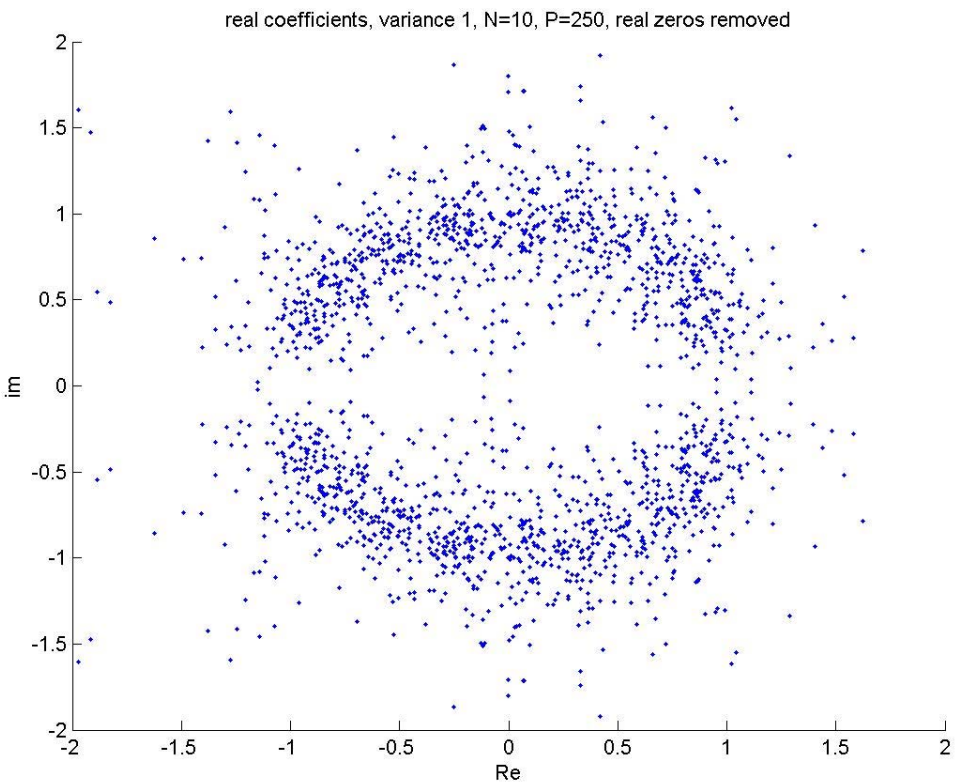
Complex zeros of Example R1

$$f_N(z) = \sum_{k=0}^{N} a_k z^k$$

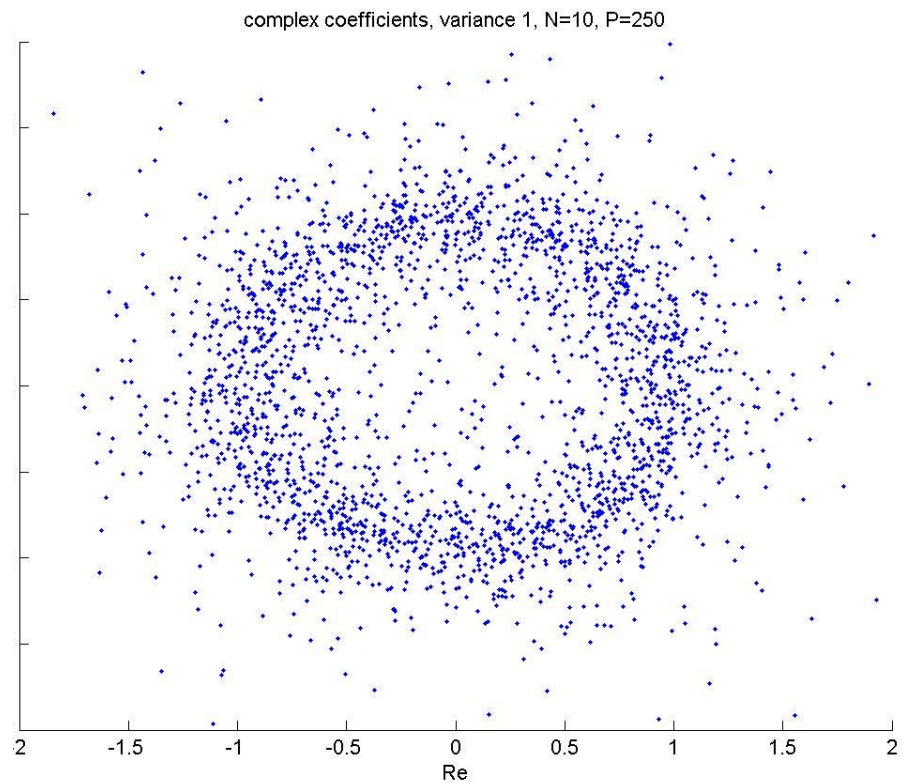
a_k is real standard Gaussian random variable

(Shepp and Vanderbei)

Example R1



Example C1

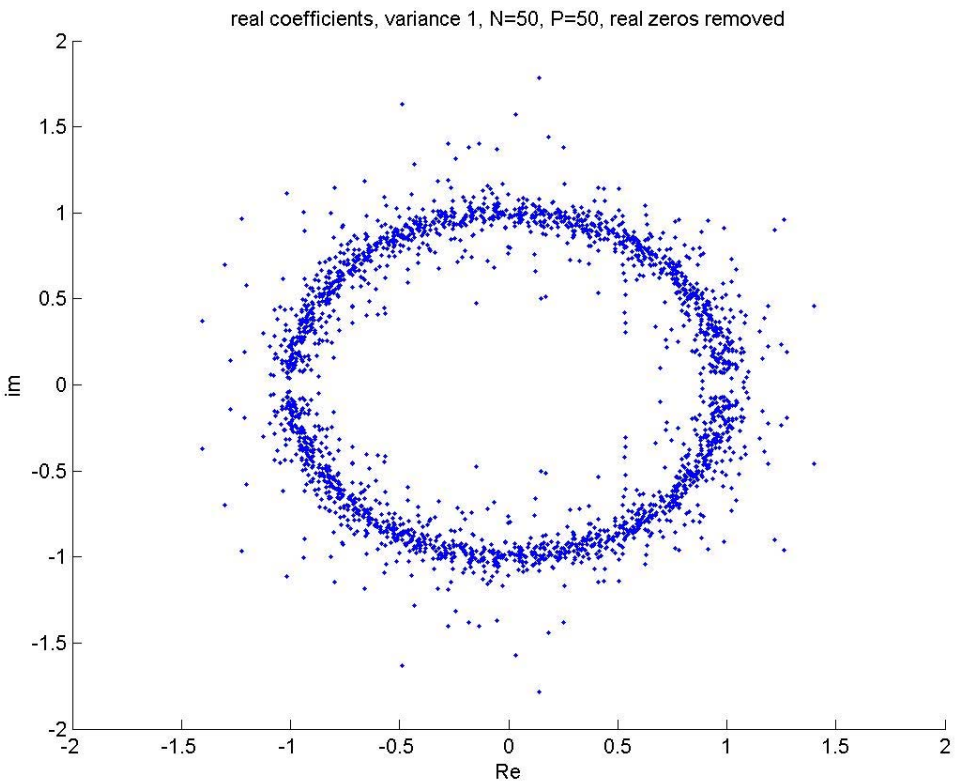


Complex Zeros of Example R1

$$f_N(z) = \sum_{k=0}^{N} a_k z^k$$

a_k is real standard Gaussian random variable

Example R1



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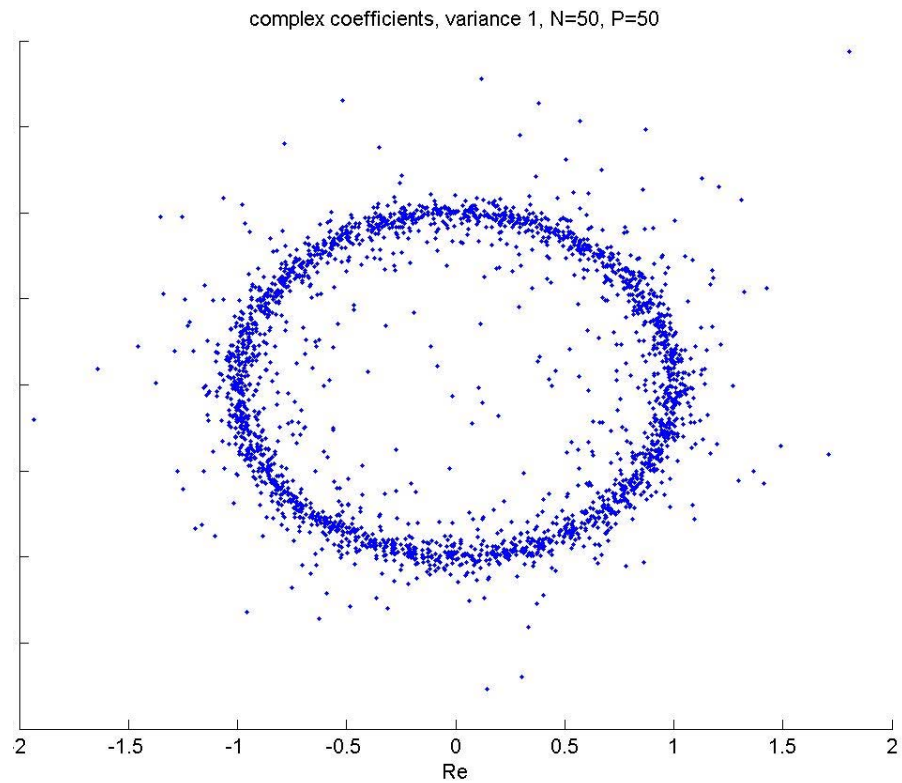


Complex Zeros of Example R2

$$f_N(z) = \sum_{k=0}^{N} a_k z^k \quad a_k \text{ is real Gaussian random variable with variance } \sigma_k^2$$

(Prosen)

But he assumed more general coefficients

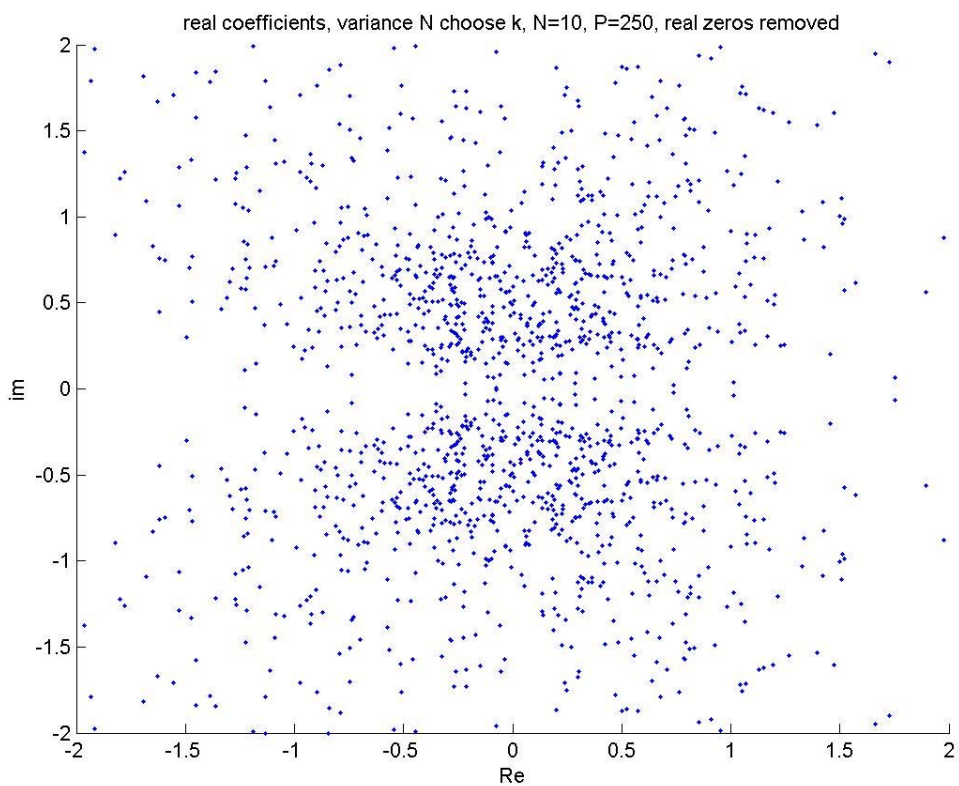
One choice is $\sigma_k^2 = \binom{N}{k}$, which corresponds to Example R2:

$$f_N(z) = \sum_{k=0}^{N} a_k \binom{N}{k}^{\frac{1}{2}} z^k \quad a_k \text{ is complex standard Gaussian random variable}$$

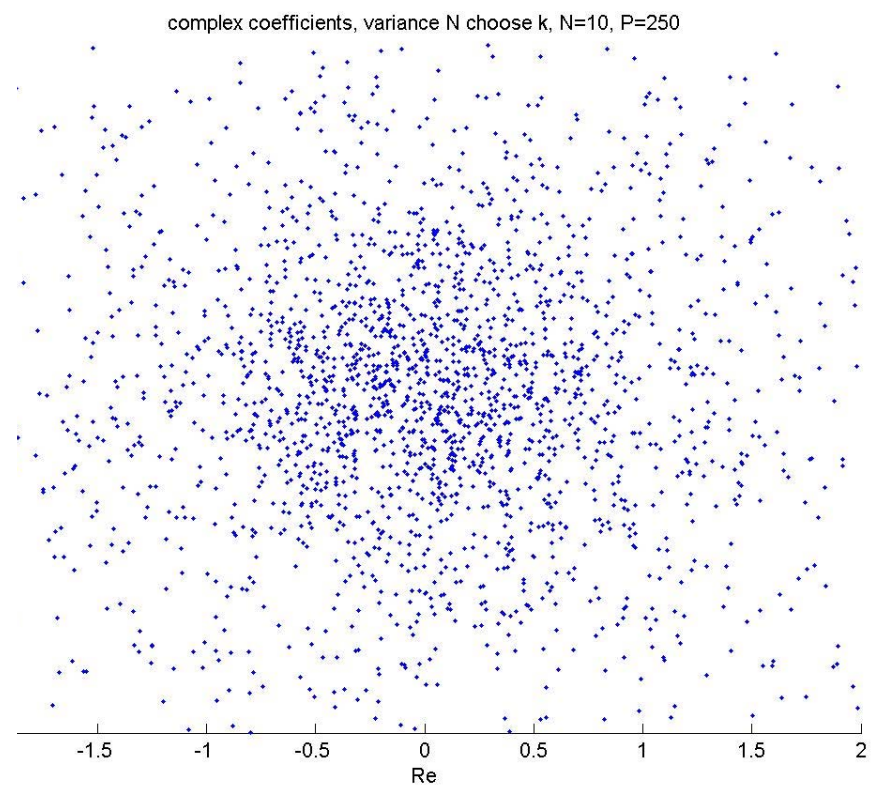
Example R2

$$f_N(z) = \sum_{k=0}^{N} a_k \binom{N}{k}^{\frac{1}{2}} z^k \qquad a_k \text{ is real standard Gaussian random variable}$$

Example R2



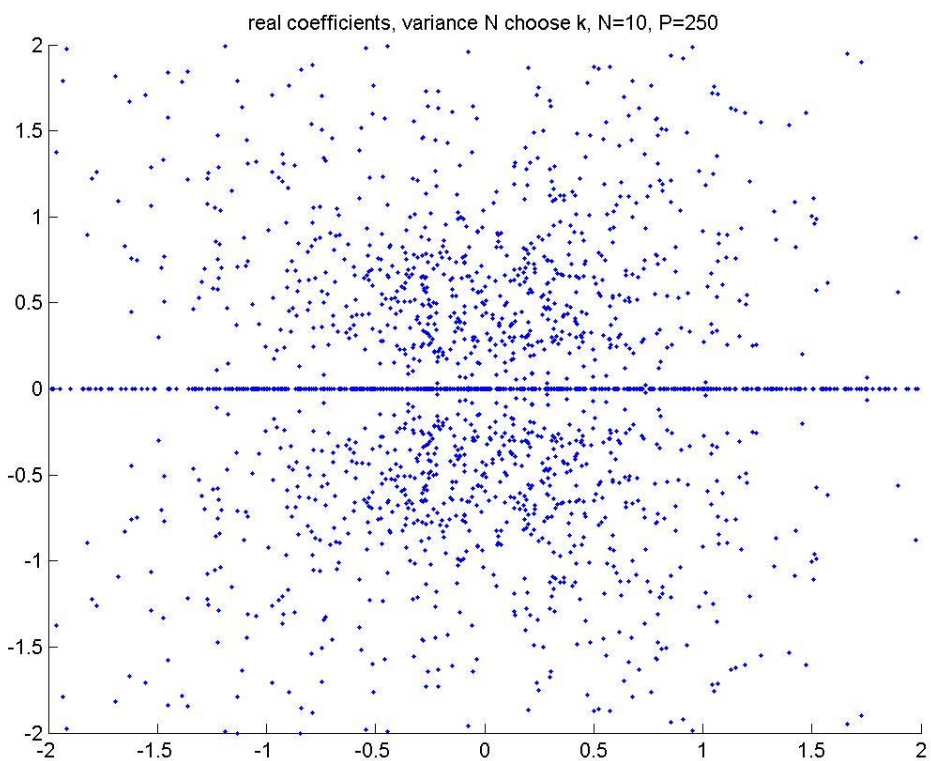
Example C2

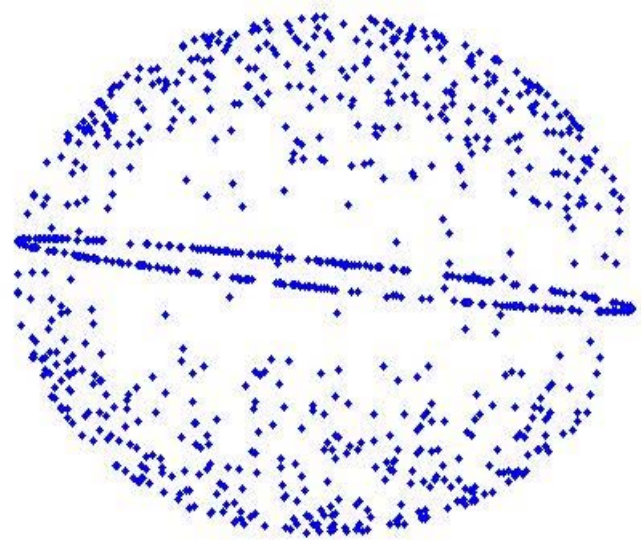


Example R2 Plotting on Sphere

$$f_N(z) = \sum_{k=0}^{N} a_k \binom{N}{k}^{\frac{1}{2}} z^k \qquad a_k \text{ is real standard Gaussian random variable}$$

Example R2





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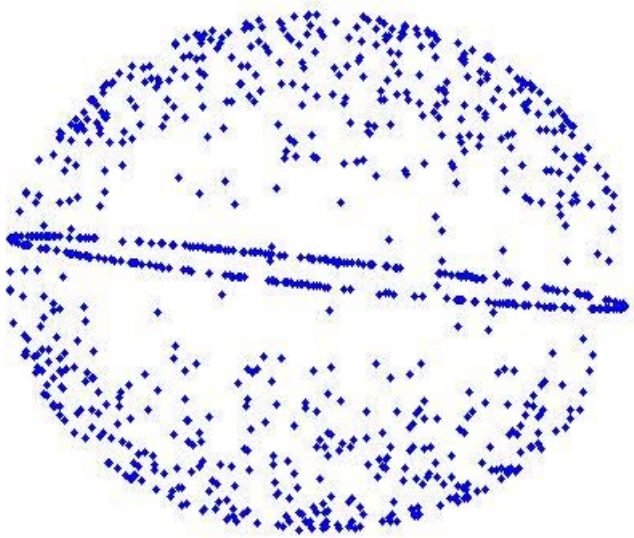
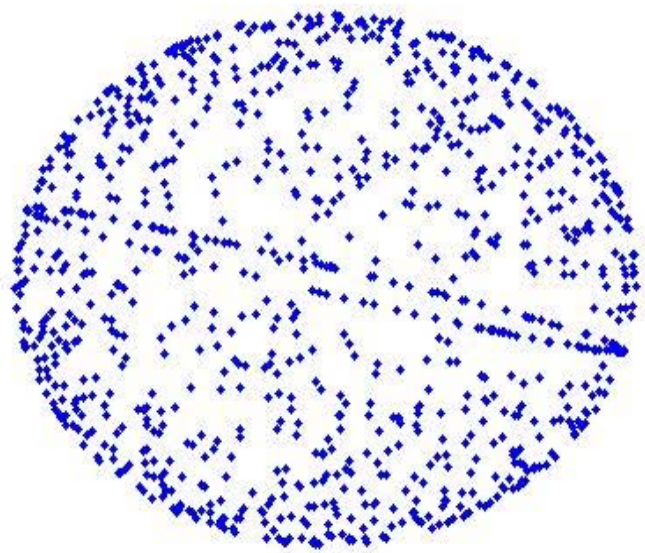


$N = 50$

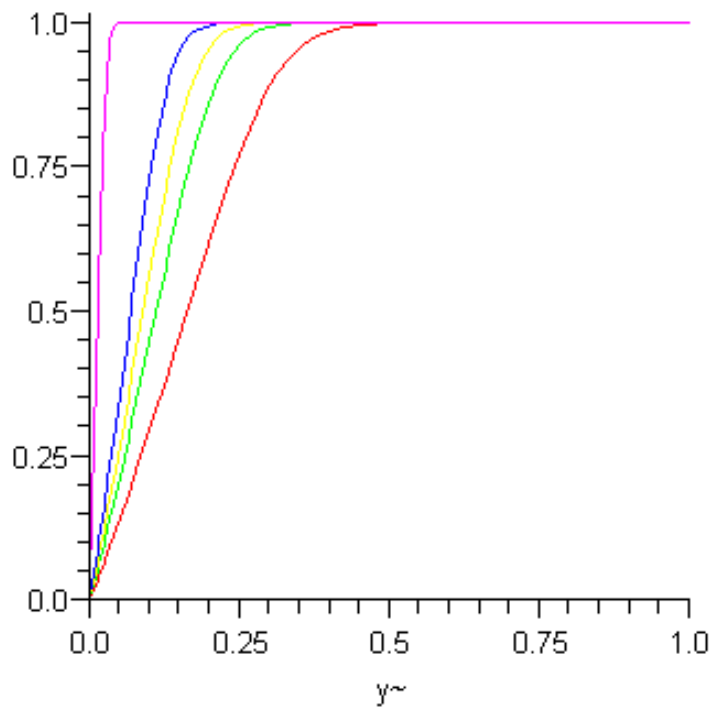


Example

$$f_N(z) = \sum_{k=0}^{N} a_k \binom{N}{k}^{\frac{1}{2}} z^k$$

a_k is real standard Gaussian random variable

$$E(Z_{f_N(iy)})$$



Notation

$$f_N(z) = \sum_{k=0}^{N} c_k \binom{N}{k}^{\frac{1}{2}} z^k$$

c_k is complex Gaussian random variable with associated measure $\mathrm{d}\gamma$

Let $c = (c_0, \ldots, c_N) = a + ib$. Consider two special cases:

1. $d\gamma_{cx} = \dfrac{1}{\pi^{N+1}} e^{-||c||^2} dc, c \in \mathbb{C}^{N+1}$ (standard complex Gaussian case)
2. $d\gamma_{real} = \dfrac{1}{(2\pi)^{(N+1)/2}} e^{-||a||^2/2} da, a \in \mathbb{R}^{N+1}$ (standard real Gaussian case)

$$f_N(z) = \sum_{k=0}^{N} c_k \binom{N}{k}^{\frac{1}{2}} z^k = \sum_{k=0}^{N} a_k \binom{N}{k}^{\frac{1}{2}} z^k$$

Result in One Variable

$$f_N(z) = \sum_{k=0}^{N} c_k \binom{N}{k}^{\frac{1}{2}} z^k$$

c_k is complex Gaussian random variable with associated measure $\mathrm{d}\gamma$

For z away from the real line,

$$E_{\gamma_{real}}(Z_{f_N(z)}) - E_{\gamma_{cx}}(Z_{f_N(z)}) \to 0, \text{ as } N \to \infty$$

Lemma: For z away from the real line,

$$E_{\gamma_{real}}(Z_{f_N(z)}) = E_{\gamma_{cx}}(Z_{f_N(z)}) + O(e^{-\lambda_z N})$$

for some $\lambda_z > 0$.

PF: Two of the main ingredients:

1. Poincare-Lelong formula $\quad E(Z_f) = E(\frac{i}{\pi}\partial\bar{\partial}\log|f|)$

2. Asymptotics of the normalized Szego kernel $\quad \frac{(1+z^2)^N}{(1+|z|^2)^N}$

Scaling Limit in One Variable

$$f_N(z) = \sum_{k=0}^{N} c_k \binom{N}{k}^{\frac{1}{2}} z^k$$

c_k is complex Gaussian random variable with associated measure $\mathrm{d}\gamma$

Prosen: Let $K_\gamma^\infty(y) = \lim_{N\to\infty} \frac{1}{N} E_\gamma(Z_{f_N(\frac{z}{\sqrt{N}})})$.

(Note: it is a function of only y.)

Then

$$K_{\gamma_{real}}^\infty(y) \to K_{\gamma_{cx}}^\infty(y)$$

as $|y| \to \infty$, and for y near 0,

$$K_{\gamma_{real}}^\infty(y) = O(|y|)$$

Note: Prosen assumed more general Gaussian coefficients, not just these.

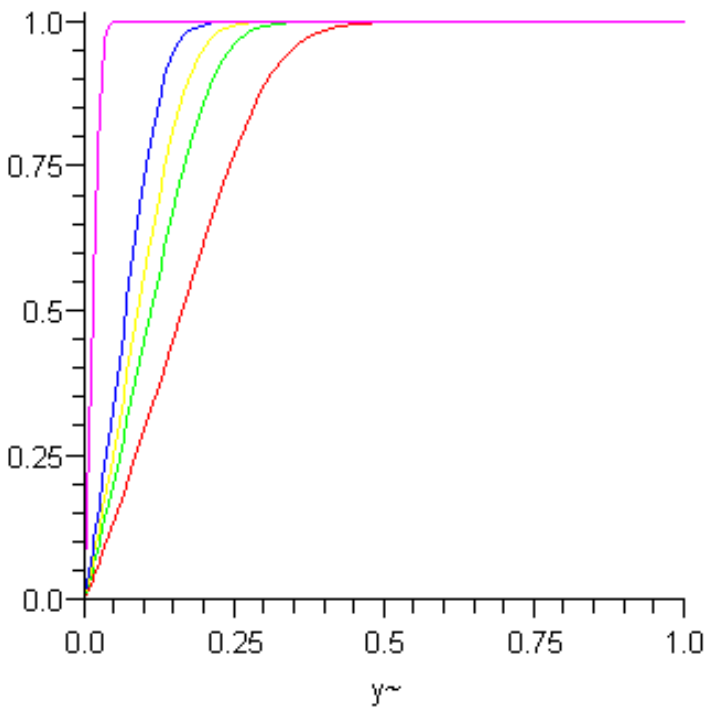
Scaling limit

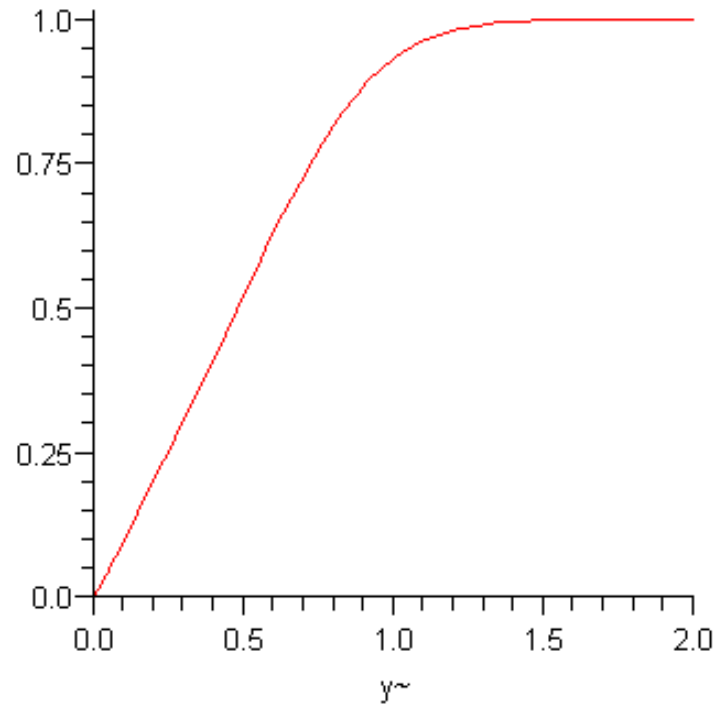
$$f_N(z) = \sum_{k=0}^{N} a_k \binom{N}{k}^{\frac{1}{2}} z^k$$

a_k is real standard Gaussian random variable

$$E(Z_{f_N(iy)})$$

$$\lim_{N\to\infty} \frac{1}{N} E\left(Z_{f_N(\frac{iy}{\sqrt{N}})}\right)$$

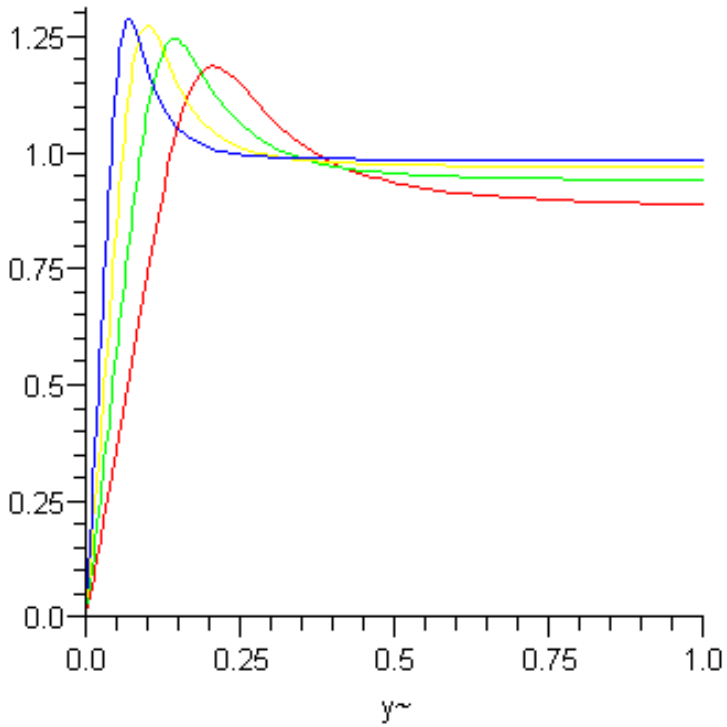




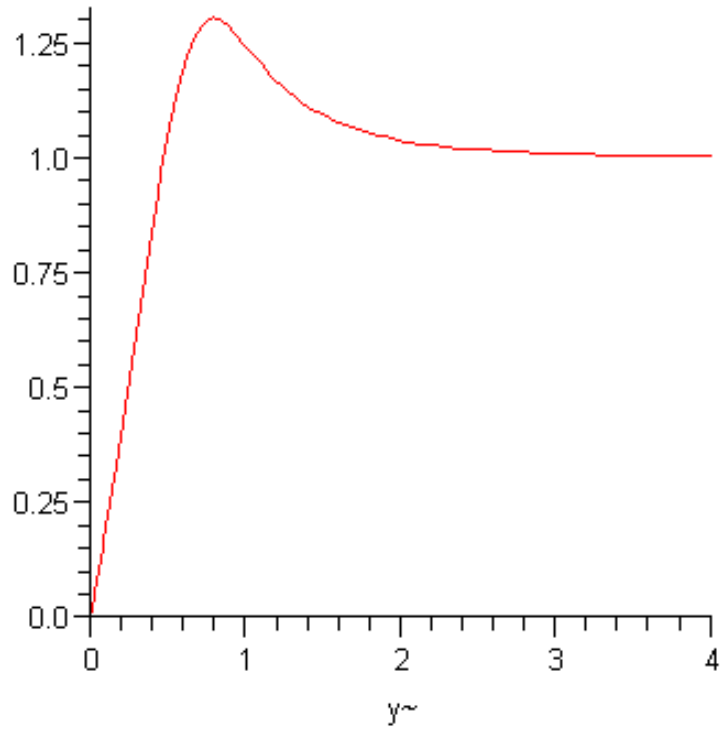
Critical Points in One Variable

$$E_{\gamma_{real}}(C_{f_N}(z)) = E_{\gamma_{cx}}(C_{f_N}(z)) + O(e^{-\lambda_z N}), z \in \mathbb{C}\backslash\mathbb{R}$$

$E(C_{f_N}(iy))$



(Scaling limit)



Systems of Polynomials

$$h_{m,N} = (f_{1,N}, ..., f_{m,N}) : \mathbb{C}^m \to \mathbb{C}^m,$$

$f_{q,N}$ is a polynomial of the form

$$f_{q,N}(z) = \sum_{|J|=0}^{N} c_J^q \binom{N}{J}^{1/2} z^J$$

c_J^q is a complex random variable with associated measure γ.

$$\begin{aligned}
z &= (z_1, ..., z_m) \\
|J| &= j_1 + \cdots + j_m \\
c_J^q &= c_{j_1 \cdots j_m}^q \in \mathbb{C} \\
\binom{N}{J} &= \binom{N}{j_1, ..., j_m} = \frac{N!}{(N - j_1 - ... - j_m)! j_1! \ ... \ j_m!} \\
z^J &= z_1^{j_1} ... z_m^{j_m}.
\end{aligned}$$

Note: The c_J^q's are different for each f_q.

Main Theorem 1

Theorem 1: Let λ_z be a positive constant. Then

$$E_{\gamma_{real}}(Z_{h_N(z)}) = E_{\gamma_{cx}}(Z_{h_N(z)}) + O(e^{-\lambda_z N}),$$

for all $z \in \mathbb{C}^m \backslash \mathbb{R}^m$.

Why?

1. Use independence

$$\left(\frac{i}{\pi}\right)^m E\Big(\partial\bar{\partial}\log|f^1| \wedge ... \wedge \partial\bar{\partial}\log|f^m|\Big)$$

$$= \left(\frac{i}{\pi}\right)^m E\Big(\partial\bar{\partial}\log|f^1|\Big) \wedge ... \wedge E\Big(\partial\bar{\partial}\log|f^m|\Big)$$

2. Use one variable case

Main Theorem 2

Theorem 2: Consider $K^{\infty}_{\gamma_{real}}(z) = \lim_{N\to\infty} \frac{1}{N} E_{\gamma_{real}}(Z_{h_N(\frac{z}{\sqrt{N}})})$

We have that, for y near 0,

$$K^{\infty}_{\gamma_{real}}(y) = O(||y||), \quad m = 1$$
$$K^{\infty}_{\gamma_{real}}(y) = O(\frac{1}{||y||^m}), \; m \geq 2$$

Why?

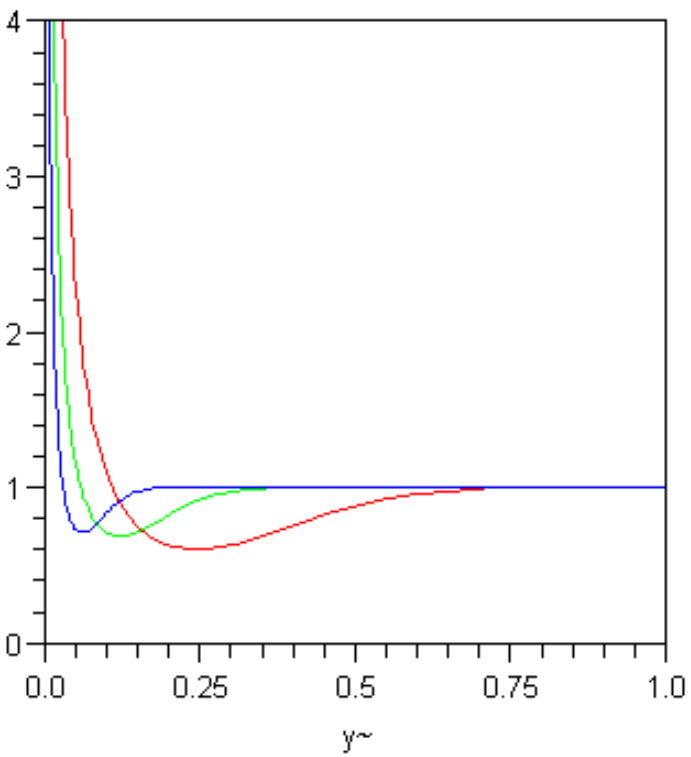
$$\frac{\partial^2}{\partial y^2}|y| = 0, \qquad \text{for } y \neq 0 \in \mathbb{R},$$
$$\frac{\partial^2}{\partial y_j \partial y_k}||y|| = O(\frac{1}{||y||}), \; \text{for } y \neq 0 \in \mathbb{R}^m, m \geq 2.$$

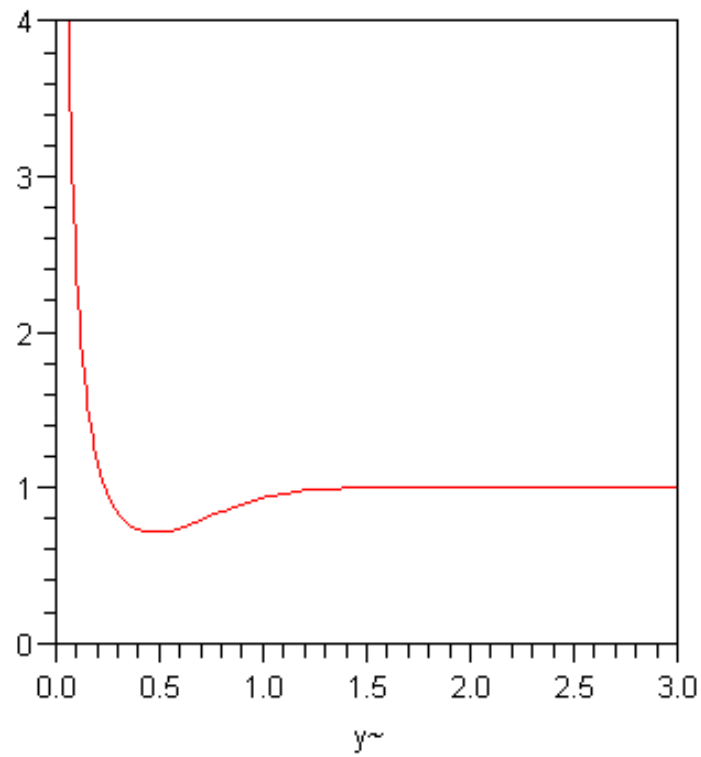
A Picture

(for the two variables case)

$$E_{\gamma_{real}}(Z_{h_N(z,w)})\Big|_{(z,w)=(iy,0)}$$

(Scaling limit)





Critical Points in Several Variables

$$h_{m,N} : \mathbb{C}^m \to \mathbb{C}$$

$h_{m,N}(z)$ is a complex random polynomial of the form

$$h_{m,N}(z) = \sum_{|J|=0}^{N} c_J \binom{N}{J}^{1/2} z^J$$

Instead: zeros of $(f_{1,N}, ..., f_{m,N}) : \mathbb{C}^m \to \mathbb{C}^m$

$f_{q,N}(z)'$s are the derivatives of $h_{m,N}(z)$:

$$f_{q,N}(z) = \frac{\partial h_{m,N}}{\partial z_q} = \sum_{|J|=0}^{N} c_J \binom{N}{J}^{1/2} \frac{\partial}{\partial z_q} z^J$$

Note: c'_Js are the same for each f_q.

Main Theorem 3

(Theorem 1 for critical points)

Let λ_z be a positive constant, and let γ_{cx}, and γ_{real} be defined as before. We have

$$E_{\gamma_{real}}(C_{h_N(z)}) = E_{\gamma_{cx}}(C_{h_N(z)}) + O(e^{-\lambda_z N}),$$

for all $z \in \mathbb{C}^m \backslash \mathbb{R}^m$.

Proof: Two of the Main Ingredients:

1. *Real* Kac-Rice Formula $\qquad E_{\gamma_{real}}(f_j f_k) \neq 0$

2. Asymptotics of the normalized Szego Kernel

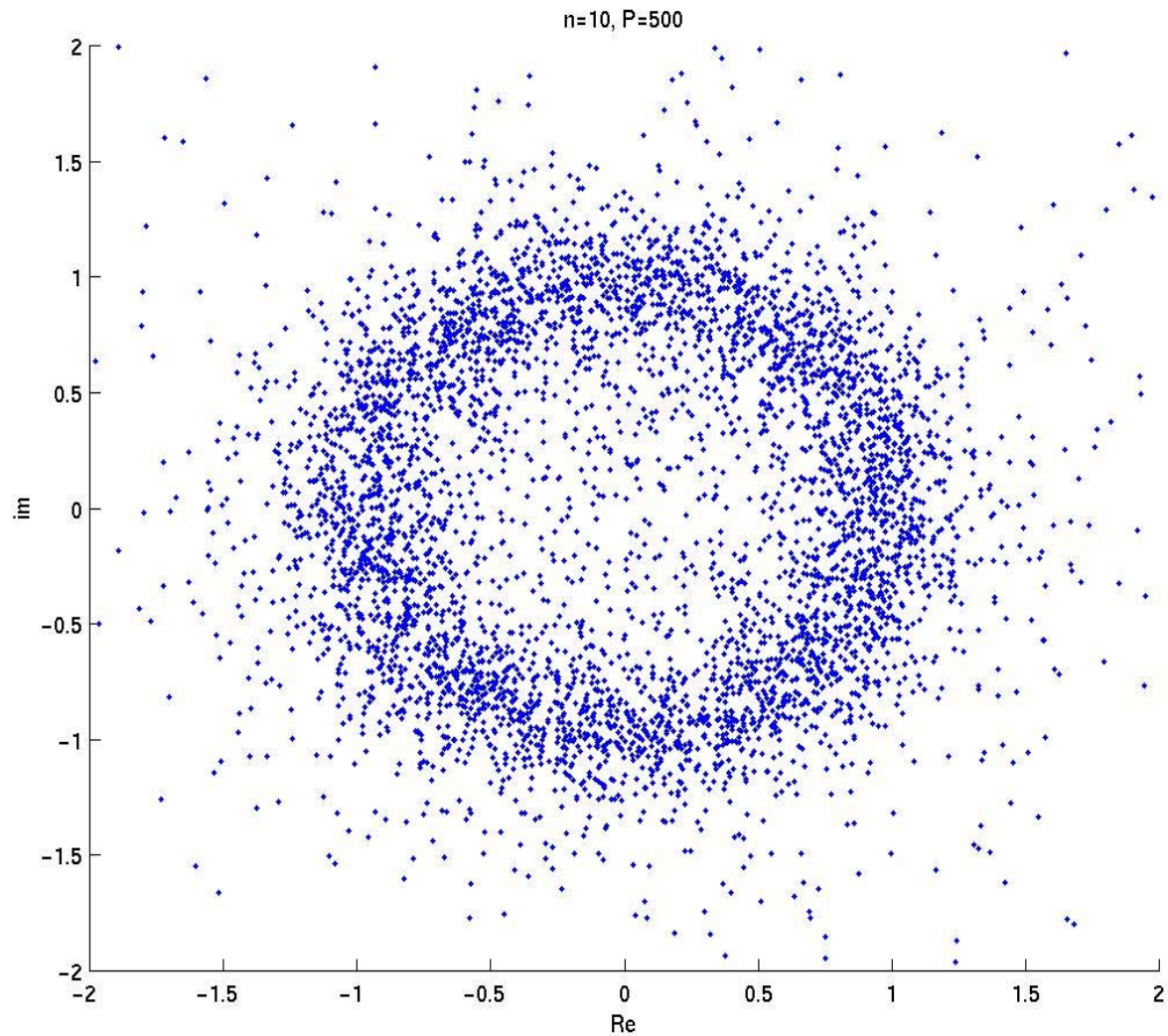
The End

Example C1

$$f_N(z) = \sum_{k=0}^{N} c_k z^k$$

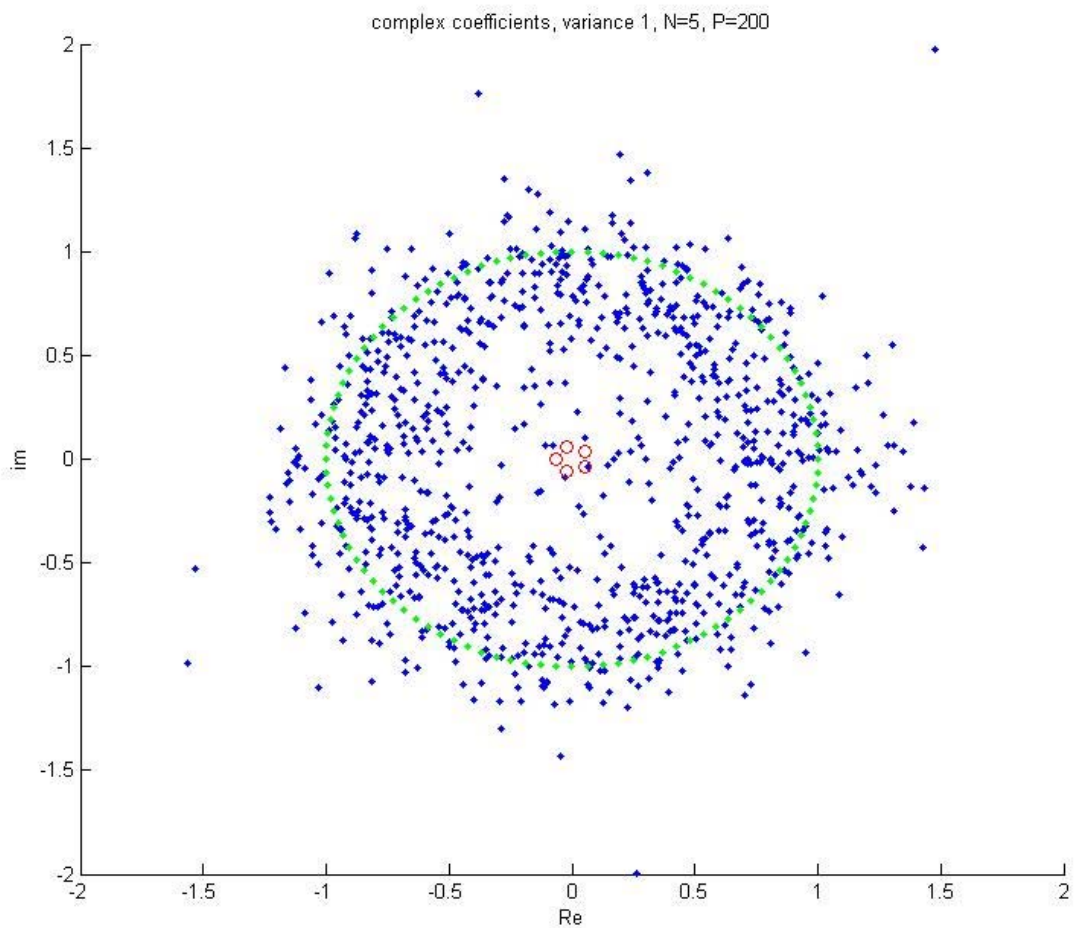
c_k is complex standard Gaussian random variable

$N = 10$
$\#$ of polynomials $= 500$
Total number of zeros $= 5000$



Why the unit circle?

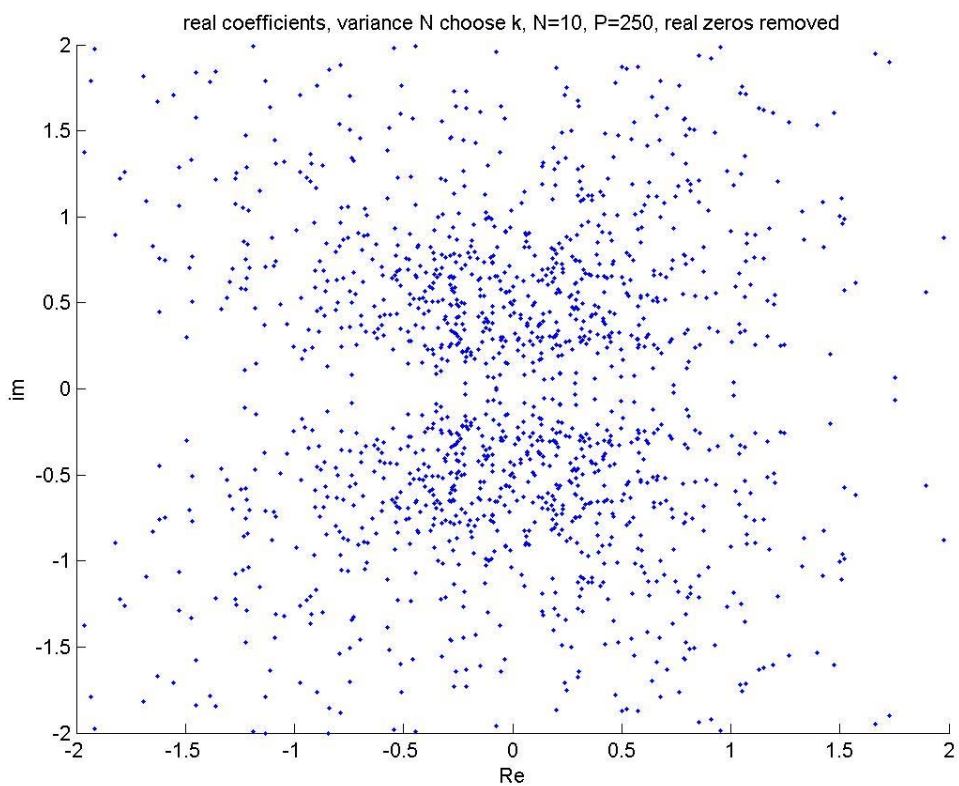
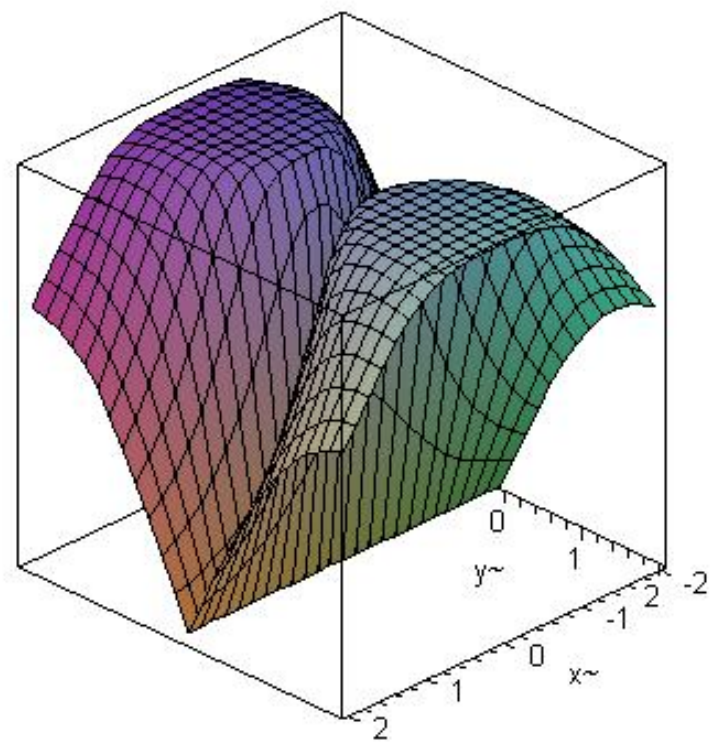
- $\tilde{f}_N(z) = \sum_{k=0}^{N} \tilde{c_k} z^k$ $\quad$ $\tilde{c_k}$ is complex Gaussian mean μ_k, variance 1.
- $g_N(z) = \sum_{k=0}^{N} \mu_k z^k$ $\quad$ (not random) $\quad$ Take $\mu_k \to 0$.



Example R2

$$f_N(z) = \sum_{k=0}^{N} a_k \binom{N}{k}^{\frac{1}{2}} z^k$$

a_k is real standard Gaussian random variable

$$E(Z_{f_N(z)})$$

Example R2